Neveu-Schwarz and operators algebras III
Subfactors and Connes fusion

Sébastien Palcoux

Abstract

This paper is the third of a series giving a self-contained way from the Neveu-Schwarz algebra to a new series of irreducible subfactors. Here we introduce the local von Neumann algebra of the Neveu-Schwarz algebra, to obtain Jones-Wassermann subfactors for each representation of the discrete series. Then using primary fields we prove the irreducibility of these subfactors; to next compute the Connes fusion ring and obtain the explicit formula of the subfactors indices.

Contents

1 Introduction **2**

2 Local von Neumann algebras **9**

3 Primary fields **26**

4 Connes fusion and subfactors **42**

1 Introduction

1.1 Background of the series

In the 90's, V. Jones and A. Wassermann started a program whose goal is to understand the unitary conformal field theory from the point of view of operator algebras (see [7], [24]). In [25], Wassermann defines and computes the Connes fusion of the irreducible positive energy representations of the loop group $LSU(n)$ at fixed level ℓ, using primary fields, and with consequences in the theory of subfactors. In [19] V. Toledano Laredo proves the Connes fusion rules for $LSpin(2n)$ using similar methods. Now, let $\mathrm{Diff}(\mathbb{S}^1)$ be the diffeomorphism group on the circle, its Lie algebra is the Witt algebra $\mathfrak{W}$ generated by d_n ($n \in \mathbb{Z}$), with $[d_m, d_n] = (m-n)d_{m+n}$. It admits a unique central extension called the Virasoro algebra $\mathfrak{Vir}$. Its unitary positive energy representation theory and the character formulas can be deduced by a so-called Goddard-Kent-Olive (GKO) coset construction from the theory of $LSU(2)$ and the Kac-Weyl formulas (see [26], [3], [26]). In [11], T. Loke uses the coset construction to compute the Connes fusion for $\mathfrak{Vir}$. Now, the Witt algebra admits two supersymmetric extensions $\mathfrak{W}_0$ and $\mathfrak{W}_{1/2}$ with central extensions called the Ramond and the Neveu-Schwarz algebras, noted $\mathfrak{Vir}_0$ and $\mathfrak{Vir}_{1/2}$. In this series ([13], [14] and this paper), we naturally introduce $\mathfrak{Vir}_{1/2}$ in the vertex superalgebra context of $L\mathfrak{sl}_2$, we give a complete proof of the classification of its unitary positive energy representations, we obtain directly their character; then we give the Connes fusion rules, and an irreducible finite depth type II_1 subfactors for each representation of the discrete series. Note that we could do the same for the Ramond algebra $\mathfrak{Vir}_0$, using twisted vertex module over the vertex operator algebra of the Neveu-Schwarz algebra $\mathfrak{Vir}_{1/2}$, as R. W. Verrill [23] and Wassermann [27] do for twisted loop groups.

1.2 Overview of the paper

Now, $\widehat{\mathfrak{g}}$ and $\mathfrak{Vir}_{1/2}$ give local superalgebras $\widehat{\mathfrak{g}}(I)$ and $\mathfrak{Vir}_{1/2}(I)$ by smearing with the smooth functions vanishing outside of I a proper interval of $\mathbb{S}^1$. By Sobolev estimates, the action on the positive energy representations is continuous. We generate their von Neumann algebra, included in an algebra of fermions. By Takesaki devissage and coset construction, we obtain that these algebras are the hyperfinite III_1 factor, whose the supercommutants are generated by chains of compressed fermions. Also, there is Haag-Araki duality on the vacuum, and outside, a Jones-Wassermann subfactor as a failure of duality.

The compressed fermions are examples of primary fields. We construct them in general from maps intertwining two irreducible representations, dealing with spaces of densities. We see that these maps are completely characterized, bounded and classified by coset for two particular charges α, β. We obtain also their braiding relations, which allow to give the leading term of a kind of OPE for smeared primary fields, which permit, to have the von Neumann density and the irreducibility of the subfactors.

Then, we obtain irreducible bimodules of local von Neumann algebras, giving the framework to define the Connes fusion. Its rules are a direct consequence of the transport formula (explaining the intertwining for chains), which is proved by the braiding relations and the von Neumann density. The rules give the dimension of the space of primary fields, they show also that the subfactors are finite index, explicitly given by the square of the quantum dimension, a fusion ring character given as unique positive eigenvalue of a fusion matrix, and a product of two quantum dimensions of $LSU(2)$ by Perron-Frobenius theorem.

1.3 Main results

Let $p = 2i+1$, $q = 2j+1$ and $m = \ell+2$, we note H^{ℓ}_{ij} the L^2-completion of $L(c_m, h^m_{pq})$. We define the Connes fusion $\boxtimes$ on the discrete series representations of charge c_m, as bimodules of the hyperfinite III_1-factor generated by the local Neveu-Schwarz algebra $\mathfrak{Vir}_{1/2}(I)$, with I a proper interval of $\mathbb{S}^1$.

Theorem 1.1. *(Connes fusion)*

$$H^{\ell}_{ij} \boxtimes H^{\ell}_{i'j'} = \bigoplus_{(i'',\, j'') \in \langle i,i' \rangle_{\ell} \times \langle j,j' \rangle_{\ell+2}} H^{\ell}_{i''j''}$$

with $\langle a, b \rangle_n = \{c = |a-b|, |a-b|+1, ..., a+b \mid a+b+c \leq n\}$.

Let $\mathcal{M}^{\ell}_{ij}(I)$ be the von Neumann algebra generated on H^{ℓ}_{ij}, by the bounded function of the self-adjoint operators of $\mathfrak{Vir}_{1/2}(I)$.

Theorem 1.2. *(Haag-Araki duality on the vacuum)*

$$\mathcal{M}_{00}^{\ell}(I) = \mathcal{M}_{00}^{\ell}(I^c)^{\natural}$$

with $X^{\natural}$ be the supercommutant of X.

As a failure of Haag-Araki duality out of the vacuum, we have:

Theorem 1.3. *(Jones-Wassermann subfactor)*

$$\mathcal{M}_{ij}^{\ell}(I) \subset \mathcal{M}_{ij}^{\ell}(I^c)^{\natural}$$

It's a finite depth, irreducible, hyperfinite III_1-subfactor, isomorphic to the hyperfinite III_1-factor $\mathcal{R}_\infty$ tensor the II_1-subfactor :

$$(\bigcup \mathbb{C} \otimes End_{\mathfrak{Vir}_{1/2}}(H_{ij}^{\ell})^{\boxtimes n})'' \subset (\bigcup End_{\mathfrak{Vir}_{1/2}}(H_{ij}^{\ell})^{\boxtimes n+1})''$$

of index $\frac{sin^2(p\pi/m)}{sin^2(\pi/m)}.\frac{sin^2(q\pi/(m+2))}{sin^2(\pi/(m+2))}$, *with* $p = 2i+1$, $q = 2j+1$, $m = \ell + 2$.

1.4 Local von Neumann algebras

For the loop algebra $L\mathfrak{g}$ and the Virasoro algebra $\mathfrak{Vir}$, we can work with the cooresponding groups: LG and Diff($\mathbb{S}^1$). For the Neveu-Schwarz algebra, there is no group corresponding to the supergenerators G_r, and so we need to work with unbounded operators. From the $\mathfrak{g}$-supersymmetric algebra $\widehat{\mathfrak{g}}$, we build a local Lie superalgebra $\widehat{\mathfrak{g}}(I)$, with I a proper interval of $\mathbb{S}^1$, by smearing with the smooth functions vanishing outside of I. In the same way, we define the local Neveu-Schwarz Lie superalgebra $\mathfrak{Vir}_{1/2}(I)$. Thanks to Sobolev estimates, these local algebras (containing unbounded operators) are represented continuously on the L_0-smooth completion of their positive energy representation. Now, we define the von Neumann algebras generated by these local algebras as the von Neumann algebra generated by the bounded functions of our self-adjoint operators; they are $\mathbb{Z}_2$-graded von Neumann algebras. Now, $\widehat{\mathfrak{g}}$ acts on a complex and real fermionic Fock space which decomposes into all its irreducible positive energy representations (with multiplicity spaces), and by coset construction we can do the same with $\mathfrak{Vir}_{1/2}$. Then, we see that the previous von Neumann algebras are included with conditional expectation in a big von Neumann algebra $\mathcal{M}(I)$ generated by smeared real and complex fermions, which is known (by [25] and a doubling construction) to be the hyperfinite III_1 factor; now, the modular action is ergodic, so by Takesaki devissage, $\mathcal{N}(I) = \pi(\mathfrak{Vir}_{1/2}(I))''$ is also the hyperfinite III_1 factor, and by the definition of type III, so is for every subrepresentations, so in particular for $\pi_i(\mathfrak{Vir}_{1/2}(I))''$, with π_i a generic irreducible positive energy representation. We deduce local equivalence, ie, the discrete series representations are unitary equivalent when they are restricted to $\mathfrak{Vir}_{1/2}(I)$; we deduce also Haag-Araki duality:

$$\pi_0(\mathfrak{Vir}_{1/2}(I^c))^\natural = \pi_0(\mathfrak{Vir}_{1/2}(I))''$$

with $X^\natural$ the supercommutant of X, from the known Haag-Araki duality of $\mathcal{M}(I)$, because the vacuum vector of H_0 is invariant by the modular operator $\triangle$ of $\mathcal{M}(I)$. Outside of the vacuum, we have a Jones-Wassermann subfactor:

$$\pi_i(\mathfrak{Vir}_{1/2}(I))'' \subset \pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural$$

as a failure of Haag-Araki duality.

1.5 Primary fields

Let p_0 be the projection on the vacuum representation H_0. The Jones relation $p_0\mathcal{M}(I)p_0 = \mathcal{N}(I)p_0$, implies that $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ is generated by products of compressed real and complex fermions: $p_0\psi_1(f_1)p_{i_1}\psi_2(f_2)p_{i_2}...\psi_n(f_n)p_0$, with p_i the projection on $H_i \subset H$ and f_s localized in I. The $p_i\psi(f)p_j$ are bounded operators intertwining the action of $\mathfrak{Vir}_{1/2}(I^c)$ between the representations H_i and H_j. We want to interpret these compressions as smeared primary fields. We define a primary field as a linear operator:

$$\phi^k_{ij} : H_j \otimes \mathcal{F}^\sigma_{\lambda,\mu} \to H_i$$

that superintertwines the action of $\mathfrak{Vir}_{1/2}$; with H_i, $H_j$on the discrete series of $\mathfrak{Vir}_{1/2}$ (k is called the charge of ϕ^k_{ij}), and $\mathcal{F}^\sigma_{\lambda,\mu}$ an ordinary representation of $\mathfrak{Vir}_{1/2}$ with base $(v_i)_{i\in\mathbb{Z}+\frac{\sigma}{2}}$, $(w_j)_{j\in\mathbb{Z}+\frac{1-\sigma}{2}}$, and:

(a) $L_n.v_i = -(i+\mu+\lambda n)v_{i+n}$
(b) $G_s.v_i = w_{i+s}$
(c) $L_n.w_j = -(j+\mu+(\lambda-\frac{1}{2})n)w_{j+n}$
(d) $G_s.w_j = -(j+\mu+(2\lambda-1)s)v_{j+s}$
with $\lambda = 1-h_k$, $\mu = h_j - h_i$, $\sigma = 0, 1$.

Let the space of densities $\{f(\theta)e^{i\mu\theta}(d\theta)^{\lambda} | f \in C^{\infty}(\mathbb{S}^1)\}$ where a finite covering of Diff($\mathbb{S}^1$) acts by reparametrisation $\theta \to \rho^{-1}(\theta)$ (if $\mu \in \mathbb{Q}$). Then its Lie algebra acts on too, so that it's a $\mathfrak{Vir}$-module vanishing the center. Finally, an equivalent construction with superdensities gives a model for $\mathcal{F}^{\sigma}_{\lambda,\mu}$ as $\mathfrak{Vir}_{1/2}$-module.

This primary field is equivalent to general vertex operators $\phi^{k}_{ij}(z)$ (called the ordinary part) and $\theta^{k}_{ij}(z) = [G_{-1/2}, \phi^{k}_{ij}(z)]$ (called the super part), and we prove that for i, j, k and σ fixed, such operators are completeley characterized by some compatibility conditions, so the space of primary fields associated is at most one dimensional. Note that $\sigma = 0$ gives ϕ^{k}_{ij} integer moded and $\sigma = 1$, half-integer moded. For charge $\alpha = (1/2, 1/2)$, we build these operators in the following way (an adaptation of an idea of Loke for $\mathfrak{Vir}$ [11], simplify by A. Wassermann): we start from the GKO coset construction $\mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{i} = \bigoplus H^{\ell}_{ii'} \otimes H^{\ell+2}_{i'}$, we take the vertex primary field of $LSU(2)$ of level ℓ and spin 1/2: $I \otimes \phi^{1/2,\ell}_{ij}(z, v) : \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{j} \to \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{i}$, with $v \in V_{1/2}$ (the vector representation of $SU(2)$). Let $p_{i'}$ be the projection on the block $H^{\ell}_{ii'} \otimes H^{\ell+2}_{i'}$. By compatibility relations and unicity, $p_{i'}(I \otimes \phi^{1/2,\ell}_{ij}(z, v))p_{j'} = C.z^{r}\phi^{\alpha}_{ii'jj'}(z) \otimes \phi^{\frac{1}{2},\ell+2}_{i'j'}(z, v)$, with C a constant possibly zero and $r \in \mathbb{Q}$. Now, $I \otimes \phi^{1/2,\ell}_{ij}(z, v) = \sum_{i'j'} p_{i'}(I \otimes \phi^{1/2\ell}_{ij}(z, v))p_{j'}$, so at least one is non-zero. More precisely, we prove by an irreducibility argument that $\forall j'$, $\exists i'$ with a non-zero term, and so $\phi^{\alpha}_{ii'jj'}(z)$ non-zero. Note that the simple locality relations between non-compressed smeared fermions concentrated on disjoint intervals (ie $\psi(f)\psi(g) = -\psi(g)\psi(f)$), admit a bit more complicated equivalent after compression: the braiding relations.

Now using the same idea as Tsuchiya-Nakanishi [22], we deduce the braiding relations for $\mathfrak{Vir}_{1/2}$: its braiding matrix is the braiding matrix for $LSU(2)$ at level ℓ, times the transposed of the inverse of the braiding matrix for $LSU(2)$ at level $\ell+2$ (it's proved by the contribution of the inverse of a gauge transformation of the Knizhnik-Zamolodchikov equation for the braiding of $LSU(2)$). Then, we obtain non-zero coefficients:

$$\phi^{\alpha\ell}_{ii'jj'}(z)\phi^{\alpha\ell}_{jj'kk'}(w) = \sum \mu_{rr'}\phi^{\alpha\ell}_{ii'rr'}(w)\phi^{\alpha\ell}_{rr'kk'}(z) \text{ with } \mu_{rr'} \neq 0.$$

Now if $\phi^{\alpha}_{ii'jj'} = 0$ and $\phi^{1/2,\ell}_{ij}$ and $\phi^{\frac{1}{2},\ell+2}_{i'j'}$ non-zero, then, the braiding relation of $\phi^{\alpha}_{ii'jj'}$ with its adjoint is zero, but produced some non-zero terms $\phi^{\alpha}_{ii'kk'}$ by the previous irreducibility argument, contradiction. Then, we see that $\phi^{\alpha}_{ii'jj'}$ is non-zero iff $\phi_{ij}^{1/2\ell}$ and $\phi_{i'j'}^{1/2,\ell+2}$ are non-zero, ie, $i' = i \pm 1/2$ and $j' = j \pm 1/2$ (up to some boundary restrictions). Now, for charge $\beta = (0,1)$ and the braiding with α, we do the same, from the Neveu-Schwarz fermion field $\psi(u,z) \otimes I$ commuting with $I \otimes \phi_{ij}^{\frac{1}{2},\ell}(v,w)$.

Next, by a convolution argument, the braiding runs also with two smeared primary fields concentrate on disjoint intervals. We deduce also that the von Neumann algebras $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ are generated by chains of primary fields. This new characterization is essential to prove the so-called von Neumann density: if I is a proper interval of $\mathbb{S}^1$ and I_1, I_2 are the intervals obtained by removing a point of I then, $\pi_i(\mathfrak{Vir}_{1/2}^{I_1})'' \vee \pi_i(\mathfrak{Vir}_{1/2}^{I_2})'' = \pi_i(\mathfrak{Vir}_{1/2}(I))''$. By local equivalence, we only need to prove it on the vacuum, on which the local algebra on I as generated by chains concentred on I. By linearity, the L^2-context, and a kind of OPE, we can separate into products of chains on I_1 and I_2. Next the von Neumann density implies the irreducibility of the Jones-Wassermann subfactor: $\pi_i(\mathfrak{Vir}_{1/2}(I))^\natural \cap \pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural = \mathbb{C}$, which significate that the representations H_i are irreducibles $\mathfrak{Vir}_{1/2}(I) \oplus \mathfrak{Vir}_{1/2}(I^c)$-modules.

1.6 Connes fusion and subfactors

Then, the discrete series representations are irreducibles bimodules over the local von Neumann algebra $\mathcal{M} = \pi_0(\mathfrak{Vir}_{1/2}(I))''$. We define a relative tensor product called Connes fusion $\boxtimes$ using a 4-points functions:
Consider the $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodule $Hom_{-\mathcal{M}}(H_0, H_i) \otimes Hom_{\mathcal{M}-}(H_0, H_j)$, we define a pre-inner product on by:

$$(x_1 \otimes y_1, x_2 \otimes y_2) = (-1)^{(\partial x_1 + \partial x_2)\partial y_2}(x_2^\star x_1 y_2^\star y_1 \Omega, \Omega)$$

The L^2-completion is also a $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodule, called the Connes fusion between H_i and H_j and noted $H_i \boxtimes H_j$. The fusion is associative.

We obtain a fusion ring for $\oplus$ and $\boxtimes$. The key tool to compute this fusion is the transport formula which gives explicitly how the chains on the vacuum representation, transform into chains on any representations through the intertwining relations. Thanks to the braiding relations known at charge α, we are able to prove the transport formula:

$$\pi_j(\bar{a}_{0\alpha}.a_{\alpha 0}) = \sum \lambda_k \bar{a}_{jk}.a_{kj} \quad \text{with } \lambda_k > 0.$$

with a_{kj} a charge α (ordinary part, so even) smeared primary field of $\mathfrak{Vir}_{1/2}$ between H_j and H_k concentrated on I, $\bar{a}_{jk} = a^{\star}_{kj}$, and $\pi_j : H_0 \to H_j$ the local equivalence. Now, $a_{\alpha 0} \in Hom_{-\mathcal{M}}(H_0, H_\alpha)$, so:

$$\|a_{\alpha 0} \otimes y\|^2 = (a^{\star}_{\alpha 0} a_{\alpha 0} y^{\star} y \Omega, \Omega) = (y^{\star} \pi_j (a^{\star}_{\alpha 0} a_{\alpha 0}) y \Omega, \Omega) = \sum \lambda_k \|a_{kj} y \Omega\|^2.$$

Then using the fact that $a_{\alpha 0}\mathcal{M}$ is dense in $Hom_{-\mathcal{M}}(H_0, H_\alpha)$ (by von Neumann density), a polarization and the irreduibility of the bimodules, we obtain a unitary map between $H_\alpha \boxtimes H_j$ and $\bigoplus_{k \in \langle \alpha, j \rangle} H_k$, with $k \in \langle \alpha, j \rangle$ iff ϕ^{α}_{jk} is a non-zero primary field. We obtain the fusion rule with α:

$$H_\alpha \boxtimes H_j = \bigoplus\nolimits_{k \in \langle \alpha, j \rangle} H_k.$$

Now, idem, with the braiding relations between charge α and β primary fields, we obtain a partial transport formula and partial fusion rules with β:

$$H_\beta \boxtimes H_j \leq \bigoplus\nolimits_{k \in \langle \beta, j \rangle} H_k.$$

But, the fusion rules with α permit to compute a character of the fusion ring called the quantum dimension (by Perron-Frobenius theorem). An easy way to comptute the quantum dimensions is to see that the fusion ring for the Neveu-Schwarz algebra at charge c_m is the tensor product of the fusion rings for the loop algebra at level ℓ and $\ell + 2$ (with $m = \ell + 2$), modulo a period two automorphism. Then the quantum dimensions for the Neveu-Schwarz algebra is a product of the two (coset corresponding) quantum dimensions for the loop algebra:

$$d(H^{\ell}_{ij}) = d(H^{\ell}_i).d(H^{\ell+2}_j) = \tfrac{sin((2i+1)\pi/(\ell+2))}{sin(\pi/(\ell+2))}.\tfrac{sin((2j+1)\pi/(\ell+4))}{sin(\pi/(\ell+4))}$$

Then the quantum dimensions show that these partial rules with β are the exact ones. Next, we see that the rules for α and β permit to compute all fusion rules. Finally, the Jones-Wassermann III_1-subfactors are isomorphic to II_1-subfactors tensor the hyperfinite III_1-factor, by H. Wenzl [29] and S. Popa [16]. These last subfactors are irreducibles, finite depth and finite index given by the square of the quantum dimensions.

2 Local von Neumann algebras

2.1 Recall on von Neumann algebras

Let H be an Hilbert space and $\mathcal{A}$ a unital $\star$-algebra of bounded operators.

Definition 2.1. *The commutant $\mathcal{A}'$ of $\mathcal{A}$ is the set of $b \in B(H)$ such that, $\forall a \in \mathcal{A}$, then $[a,b] := ab - ba = 0$*

Definition 2.2. *The weak operator topology closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ is the set of $a \in B(H)$ such that $\exists a_n \in \mathcal{A}$ with $(a_n\eta, \xi) \to (a\eta, \xi)$, $\forall \eta, \xi \in H$.*

Reminder 2.3. *(Bicommutant theorem) Let $\mathcal{M}$ be a unital $\star$-algebra, then:*

$$\mathcal{M}'' = \mathcal{M} \iff \bar{\mathcal{M}} = \mathcal{M}$$

Definition 2.4. *Such a $\mathcal{M}$ verifying one of these equivalents properties is called a von Neumann algebra.*

Definition 2.5. *A factor is a von Neumann algebra $\mathcal{M}$ with $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}$.*

Reminder 2.6. *(Murray and von Neumann theorem) The set of all the factors on H is a standard borelian space X and every von Neumann algebra $\mathcal{M}$ decompose into a direct integral of factors: $\mathcal{M} = \int_X^{\oplus} \mathcal{M}_x d\mu_x$*

Reminder 2.7. *(Murray and von Neumann's classification of factors)*
Let $\mathcal{M} \subset B(H)$ be a factor. We shall consider H as a representation of $\mathcal{M}'$. Thus subrepresentations of H correspond to projections in $\mathcal{M}$. If $p, q \in \mathcal{M}$ are projections, then pH and qH are unitarily equivalent as representations of $\mathcal{M}'$ iff there is a partial isometry $u \in \mathcal{M}$ between pH and qH; thus $u^*u = p$ and $uu^* = q$. We can immediately distinguish three mutually exclusive cases.
I. H has an irreducible subrepresentation.
II. H has no irreducible subrepresentation, but has a subrepresentation not equivalent to any proper subrepresentation of itself.
III. H has no irreducible subrepresentation and every subrepresentation is equivalent to some proper subrepresentation of itself.
We shall call $\mathcal{M}$ a factor of type I , II or III acording to the above cases.

Reminder 2.8. *The type I and II corresponds to factors admitting non-trivial trace, with only integer values on the projectors for the type I ($M_n(\mathbb{C})$ or $B(H)$), and non-integer values for the type II (factors generated by ICC groups for example). On the type III, the values are only 0 or ∞.*

Reminder 2.9. *(Tomita-Takesaki theory) We suppose the existence of a vector Ω (called vacuum vector) such that $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in H (ie Ω is cyclic and separating). Let $S : H \to H$ the closure of the antilinear map: $\star : x\Omega \to x^\star\Omega$. Then, S admits the polar decomposition $S = J\triangle^{\frac{1}{2}}$ with J antilinear unitary, and $\triangle^{\frac{1}{2}}$ positve; so that $J\mathcal{M}J = \mathcal{M}'$, $\triangle^{it}\mathcal{M}\triangle^{-it} = \mathcal{M}$ and $\sigma_t^\Omega(x) = \triangle^{it}x\triangle^{-it}$ gives the one parameter modular group action.*

Reminder 2.10. *(Radon-Nikodym theorem) Let Ω' be another vacuum vector, then there exists a Radon-Nikodym map $u_t \in \mathcal{U}(\mathcal{M})$, define such that $u_{t+s} = u_t\sigma_t^{\Omega'}(u_s)$ and $\sigma_t^{\Omega'}(x) = u_t\sigma_t^{\Omega}(x)u_t^\star$. Then, modulo $Int(\mathcal{M})$, σ_t^Ω is independant of the choice of Ω, ie, there exist an intrinsic $\delta : \mathbb{R} \to Out(\mathcal{M}) = Aut(\mathcal{M})/Int(\mathcal{M})$. On type I or II the modular action is internal, and so δ trivial. It's non-trivial for type III.*

Definition 2.11. *We can then define two invariants of $\mathcal{M}$, $T(\mathcal{M}) = ker(\delta)$ and $\mathcal{S}(\mathcal{M}) = Sp(\delta) = \bigcap Sp(\triangle_\Omega) \setminus \{0\}$ called the Connes spectrum of $\mathcal{M}$.*

Reminder 2.12. *(see [1]) Let $\mathcal{M}$ be a type III factor, then $\mathcal{S}(M) = \{1\}$, $\lambda^{\mathbb{Z}}$ or $\mathbb{R}_+^\star$, and then, $\mathcal{M}$ is called a III_0, III_λ or III_1 factor (with $0 < \lambda < 1$).*

Reminder 2.13. *Let $\mathcal{M} \neq \mathbb{C}$ be a von Neumann algebra on (H, Ω) then it's a III_1 factor if and only if the modular action (i.e. the action of $\mathbb{R}$ on $\mathcal{M}$ via σ_t^Ω) is ergodic (i.e. it fixes only the scalar operators).*

2.2 $\mathbb{Z}_2$-graded von Neumann algebras

Definition 2.14. *A $\mathbb{Z}_2$-graded von Neumann algebra $(\mathcal{M}, \tau)$ is a von Neumann algebra $\mathcal{M}$ given with a period two automorphism: $\tau \in Aut(\mathcal{M})$ and $\tau^2 = I$. Now $\forall x \in \mathcal{M}$, $x = x_0 + x_1$ with $x_0 = \frac{1}{2}(x+\tau(x))$ and $x_1 = \frac{1}{2}(x-\tau(x))$ called the even and the odd part of x. Then $\tau(x_0) = x_0$ and $\tau(x_1) = -x_1$. Hence $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$; if $a \in \mathcal{M}_{\varepsilon_1}$ and $b \in \mathcal{M}_{\varepsilon_2}$ then $a.b \in \mathcal{M}_{\varepsilon_1+\varepsilon_2}$.*

Definition 2.15. *A $\mathbb{Z}_2$-graded Hilbert space is an Hilbert space given with a period two unitary operator: $u \in \mathcal{U}(H)$ and $u^2 = I$, so that $H = H_0 \oplus H_1$, with H_0 and H_1 the eigenspaces of u for the eigenvalues 1 and -1. Let p_0 and p_1 the corresponding projection, then $u = p_0 - p_1 = 2p_0 - 1$.*

Remark 2.16. *Let $\mathcal{M}$ be a von Neumann algebra on H with Ω its cyclic, separating vector. Then a period two automorphism τ of $\mathcal{M}$ gives a period* *two unitary operator u of H by $u : x\Omega \to \tau(x)\Omega$. Conversely, a period two unitary operator u of H with $u.\Omega = \Omega$ gives $\tau \in Aut(\mathcal{M})$ by $\tau(x) = uxu$.*

Definition 2.17. *Let $x \in B(H)$, then, $\tau(x) = uxu$ defined a period two automorphism on $B(H)$. Then as for definition 2.14, $x = x_0 + x_1$.*
We see that $x_0 = p_0xp_0 + p_1xp_1$ and $x_1 = p_1xp_0 + p_0xp_1$.

Definition 2.18. *(Supercommutator)*
Let $[x, y]_\tau = [x_0, y_0] + [x_0, y_1] + [x_1, y_0] + [x_1, y_1]_+$

Remark 2.19. *A projection p is even, then, $\forall x \in B(H)$, $[x, p]_\tau = [x, p]$.*
In particular $[x, I]_\tau = [x, I] = 0$.

Definition 2.20. *The supercommutant $\mathcal{A}^\natural$ of $\mathcal{A}$ is the set of $b \in B(H)$ such that, $\forall a \in \mathcal{A}$, then $[a, b]_\tau = 0$.*

Definition 2.21. *Let $\kappa = p_0 + ip_1$ the Klein transformation.*

Remark 2.22. *κ is unitary, $\kappa^{-1} = \kappa^\star = p_0 - ip_1$ and $\kappa^2 = u$.*

Remark 2.23. *$ux_0u = x_0$, $ux_1u = -x_1$, $\kappa x_0\kappa^\star = x_0$, $\kappa x_1\kappa^\star = -iux_1$*

Lemma 2.24. *Let $\mathcal{A}$ be a von Neumann algebra $\mathbb{Z}_2$-graded for τ, then:*

$$\mathcal{A}^\natural = \kappa\mathcal{A}'\kappa^\star.$$

Proof. Let $a \in \mathcal{A}$ and $x \in B(H)$ such that $[x, a] = 0$.
By the relations of the remarks 2.23:
If x is even, then $[\kappa x\kappa^\star, a]_\tau = [x, a] = 0$.
If a is even, then $[\kappa x\kappa^\star, a]_\tau = \kappa[x, \kappa^\star a\kappa]\kappa^\star = \kappa[x, a]\kappa^\star = 0$.
Else, $[\kappa x\kappa^\star, a]_\tau = [-i\tau x, a]_+ = -i(uxa + aux) = -iu[x, a] = 0$
Then, $\kappa\mathcal{A}'\kappa^\star \subset \mathcal{A}^\natural$; idem, $\kappa^\star\mathcal{A}^\natural\kappa \subset \mathcal{A}'$; the result follows. ☐

Corollary 2.25. *$\mathcal{A}^\natural$ is unitary equivalent to $\mathcal{A}'$.*

Proof. κ is a unitary operator. ☐

Lemma 2.26. *Let $(\mathcal{A}, \tau)$ be a $\mathbb{Z}_2$-graded von Neumann algebra then:*

$$\mathcal{A}^{\natural\natural} = \mathcal{A}.$$

Proof. $\mathcal{A}^{\natural\natural} = \kappa(\kappa\mathcal{A}'\kappa^\star)'\kappa^\star = \kappa\kappa(\mathcal{A}'')\kappa^\star\kappa^\star$, because a von Neumann algebra is generated by its projections, and a projection is even, so commute with κ.
Then $\mathcal{A}^{\natural\natural} = u\mathcal{A}u = \tau(\mathcal{A}) = \mathcal{A}$. ☐

2.3 Global analysis

The generic discrete series representation $L(c_m, h^m_{pq})$ is a prehilbert space of finite level vectors, we note H^m_{pq} its L^2-completion.

Definition 2.27. *Let $s \in \mathbb{R}$, we define the Sobolev norms $\|.\|_{(s)}$ as follows:*

$$\|\xi\|_{(s)} := \|(I + L_0)^s \xi\| \quad \forall \xi \in L(c_m, h^m_{pq})$$

Remark 2.28. $((1 + L_0)^{2s}\xi, \xi) = \|\xi\|^2_{(s)}$

Proposition 2.29. *(Sobolev estimate) $\exists k_n, k_r > 0$ such that $\forall \xi \in L(c_m, h^m_{pq})$:*

(a) $\|L_n \xi\|_{(s)} \leq k_n (1 + |n|)^{|s|+3/2} \|\xi\|_{(s+1)}$

(b) $\|G_r \xi\|_{(s)} \leq k_r (1 + |r|)^{|s|+1/2} \|\xi\|_{(s+1/2)}$

Proof. (a) See Goodman-Wallach [4] (proposition 2.1 p 307).
(b)$2L_0 = G_r G_{-r} + G_{-r} G_r$. Then, $2(L_0\xi, \xi) = (G_r\xi, G_r\xi) + (G_{-r}\xi, G_{-r}\xi)$. So, $\|G_r\xi\|^2 \leq k_1 \|L_0^{1/2}\xi\|^2$ for any r. Now, it suffices to show the result for an eigenvector of L_0: $L_0\xi = \mu\xi$. We can take $r \leq \mu$ (otherwise $G_r\xi = 0$).
$\|G_r\xi\|^2_s = \|(1+L_0)^s G_r\xi\|^2 \leq (1+\mu-r)^{2s}\|G_r\xi\|^2 \leq (1+\mu-r)^{2s} k_1 \|L_0^{1/2}\xi\|^2 \leq (1+\mu-r)^{2s} k_1 \mu \|\xi\|^2 \leq \frac{(1+\mu-r)^{2s}}{(1+\mu)^{2s}} k_1 \|\xi\|^2_{s+1/2} \leq (1+|r|)^{2|s|+1} k_1 \|\xi\|^2_{s+1/2}$. ☐

Remark 2.30. *Thanks to $L_n = [G_{n-1/2}, G_{1/2}]_+$, we obtain directly the estimate $\|L_n\xi\|_{(s)} \leq k(1 + |n|)^{|2s|+1}\|\xi\|_{(s+1)}$ without Goodman-Wallach result.*

Definition 2.31. *Let $H^{m,s}_{pq}$ be the $\|.\|_s$-completion of $L(c_m, h^m_{pq})$ and:*

$$\mathcal{H}^m_{pq} = \bigcap_{s>0} H^{m,s}_{pq}$$

with the usual Fréchet topology from the norms $\|.\|_s$

Corollary 2.32. *$L(c_m, h^m_{pq})$ extends to a continuous representation of $\mathfrak{Vir}_{1/2}$ on $\mathcal{H}^m_{pq}$.*

Definition 2.33. *Let $d = -i\frac{d}{d\theta}$ the unbounded operator of $L^2(\mathbb{S}^1)$, let F be the subspace of finite Fourier series as a dense domain of d. Let $s \in \mathbb{R}$ and $\|f\|_{(s)} := \|(I + |\delta|)^s.f\|_1$ a Sobolev norm on F. Let F_s be the completion of F relative to $\|.\|_{(s)}$. Idem for $e^{i\theta/2}F$.*

Definition 2.34. *Let* $L_f = \sum a_n L_n$ *and* $G_h = \sum b_r G_r$ *such that* $f(\theta) = \sum a_n e^{in\theta}$, $h(z) = \sum b_r e^{ir\theta}$ *and* $f \in F$ *and* $h \in e^{i\theta/2}F$.

Notation 2.35. *Let* $(f,h)_{\mathbb{R}} := \frac{1}{2\pi i}\int_0^{2\pi} f(\theta)h(\theta)d\theta$, *with* $f.h \in F$

Lemma 2.36. *(Lie bracket relation)*

$$\begin{cases} [L_f, L_h] & = L_{d(f)h - fd(h)} + \frac{C}{12}((d^3 - d)(f), h)_{\mathbb{R}} \\ [G_f, L_h] & = G_{d(f)h - \frac{1}{2}fd(h))} \\ [G_f, G_h]_+ & = 2L_{fh} + \frac{C}{3}((d^2-1)(f), h)_{\mathbb{R}} \end{cases}$$

The $\star$*-structure:* $L_f^\star = L_{\bar{f}}$, $G_h^\star = G_{\bar{h}}$.

Proof. Direct by computation from proposition 2.9 of [13]. □

Proposition 2.37. *(Sobolev estimate)*
$\exists k > 0$ *such that* $\forall \xi \in H_{pq}^m$ *and* $f \in F$, $h \in e^{i\theta/2}F$*:*

(a) $\|L_f \xi\|_{(s)} \leq k\|f\|_{(|s|+3/2)}\|\xi\|_{(s+1)}$

(b) $\|G_h \xi\|_{(s)} \leq k\|h\|_{(|s|+1/2)}\|\xi\|_{(s+1/2)}$

Proof. It's immediate from proposition 2.29. □

Reminder 2.38. $\bigcap_{s>0} F_s = C^\infty(\mathbb{S}^1)$.

Corollary 2.39. *The operators* L_f *and* G_h *act continuously on* $\mathcal{H}_{pq}^m$, *with* $f \in C^\infty(\mathbb{S}^1)$ *and* $h \in e^{i\theta/2}C^\infty(\mathbb{S}^1)$.

Reminder 2.40. *Let* T *be an operator on a Hilbert space* H. *A subspace* $D(T)$ *of* H *is called a domain of* T *if* $T.D(T) \subset H$. *Then let* $\Gamma(T) = \{(x, T.x), x \in D(T)\}$ *be the graph of* T. *The operator* T *is closed if its graph* $\Gamma(T)$ *is closed in* $H \times H$. *An operator* $\tilde{T}$ *is an extension of* T *if* $\Gamma(T) \subset \Gamma(\tilde{T})$, *we write* $T \subset \tilde{T}$. *The operator* T *is closable if it admits a closed extension; let* $\bar{T}$ *be the smallest one. Then,* T *is closable iff* $\overline{\Gamma(T)}$ *is the graph of a linear operator (not always true). If* T *is densely defined, then its adjoint* $T^\star$ *is closed because its graph is an orthogonal. From now, every domain is dense in* H. *The operator* T *is symmetric or formally self-adjoint if* $T \subset T^\star$, *essentially self-adjoint if* $\bar{T} = T^\star$, *and self-adjoint if* $T = T^\star$.

Reminder 2.41. *(Glimm-Jaffe-Nelson commutator theorem [18] X.5)*
Let D be a diagonalizable, positve, compact resolving operator and X formally self-adjoint, with common dense domain. If $(D+I)^{-1}X$, $X(D+I)^{-1}$ and $(D+I)^{-1/2}[D,X](D+I)^{-1/2}$ are bounded, then X is essentially self-adjoint.

Lemma 2.42. *Let f, $h \in C^{\infty}(\mathbb{S}^1)$ and real, then, L_f and G_h act on $\mathcal{H}^m_{pq}$ as essentially self-adjoint operators.*

Proof. The function f is real, so $\bar{f} = f$, then, by the $\star$-structure and the unitarity of the action, L_f is formally self-adjoint. Now, L_0 is positive and by Sobolev estimate: $\|(L_0+I)^{-1}L_f\xi\| = \|L_f\xi\|_{(-1)} \leq k\|\xi\|_{(0)} = k\|\xi\|$, so $(L_0+I)^{-1}L_f$ is bounded. Now, $\|L_f\eta\| \leq k\|\eta\|_1 = k\|(L_0+I)\eta\|$, so taking $\xi = (L_0+I)\eta$, we find $\|L_f(L_0+I)^{-1}\xi\| \leq k\|\xi\|$. Finally, $[L_0, L_f] = L_h$ with $h(z) = -zf'(z)$, so combining the two previous tips with $\xi = (L_0+I)^{1/2}\eta$, we find $(L_0+I)^{-1/2}[L_0,L_f](L_0+I)^{-1/2}$ bounded too. We can do the same with G_h because $\|\xi\|_{(s+1/2)} \leq \|\xi\|_{(s+1)}$. Then, the result follows by reminder 2.41. ☐

Remark 2.43. *This result was already known for Diff$(\mathbb{S}^1)$ and hence the L_f. On the other hand $G_f^2 = L_{f^2} + kId$, so the essential self-adjointness follows by Nelson's theorem:*

Reminder 2.44. *(Nelson's theorem [12]) Let H be an Hilbert space, A and B be formally self-adjoint operator acting on a dense subspace $D \subset H$, such that $AB\xi = BA\xi$ $\forall \xi \in D$, and $A^2 + B^2$ essentially self-adjoint, then A, B are essentially self-adjoint, and their bounded function commute on H.*
Remark that we have the same result for supercommutation introducing κ.

Reminder 2.45. *Let T be a self-adjoint operator with $D(T)$ dense in H. There exist a finite measure space (Y,μ), a unitary operator $U : H \to L^2(Y,\mu)$ and a real function f, finite up to a null set on Y, such that, if M_f is the operator of multiplication by f, with domain $D(M_f)$, then $\nu \in D(T) \iff U\nu \in D(M_f)$, and $\forall g \in D(M_f)$, $UTU^{\star}g = fg$. Let h be a borelian function bounded on $\mathbb{R}$. The bounded operator $h(T)$ on H is defined by $h(T) = U^{\star}M_{h(f)}U$.*

2.4 Definition of local von Neumann algebras

Definition 2.46. *(Dixmier) Let H be an Hilbert space. An unbounded self-adjoint operator T is affiliated to a von Neumann algebra $\mathcal{M}$ if it satisfy one* *of the followings equivalent properties:*

(a) $\mathcal{M}$ contains all the spectral projection of T.

(b) $\mathcal{M}$ contains every bounded functions of T.

(c) $\forall u \in \mathcal{M}'$ unitary, $uD(T) = D(T)$ and $uT\xi = Tu\xi$, $\forall \xi \in D(T)$.

We note $T\,\eta\,\mathcal{M}$.

Remark 2.47. *By lemma 2.26, if $(\mathcal{M}, \tau)$ is a $\mathbb{Z}_2$-graded von Neumann algebra, we can add:*

(c') $\forall u \in \mathcal{M}^\natural$ unitary, $uD(T) = D(T)$, $uT\xi = (-1)^{\partial T \partial u} Tu\xi$, $\forall \xi \in D(T)$.

Definition 2.48. *Let I be a proper interval of $\mathbb{S}^1$.*
We define $C_I^\infty(\mathbb{S}^1)$ as the algebra of smooth functions vanishing out of I.

Definition 2.49. *Let $\mathfrak{Vir}_{1/2}(I)$ be the local Neveu-Schwarz Lie superalgebra, generated by L_f, G_f with $f \in C_I^\infty(\mathbb{S}^1)$, and C central.*

Lemma 2.50. *(Locality) $\mathfrak{Vir}_{1/2}(I)$ and $\mathfrak{Vir}_{1/2}(I^c)$ supercommute.*

Proof. By lemma 2.36, the computation of the brackets involve product of functions in $C_I^\infty(\mathbb{S}^1)$ and $C_{I^c}^\infty(\mathbb{S}^1)$, but $C_I^\infty(\mathbb{S}^1).C_{I^c}^\infty(\mathbb{S}^1) = \{0\}$. □

Definition 2.51. *Let p_0 be the projection on the space generated by the vectors of integer level, $p_1 = 1 - p_0$, $u = p_0 - p_1$ and $\tau(x) = uxu$.*

Definition 2.52. *Let the von Neumann algebra $\mathcal{N}_{pq}^m(I)$ be the minimal von Neumann subalgebra of $B(H_{pq}^m)$ such that the self-adjoint operators of $\mathfrak{Vir}_{1/2}(I)$ (i.e L_f, G_f with $f \in C_I^\infty(\mathbb{S}^1)$ real), are affiliated to it. See definition 2.46 for equivalent definitions. $(\mathcal{N}_{pq}^m(I), \tau)$ is a $\mathbb{Z}_2$-graded von Neumann algebra.*

Corollary 2.53. *(Jones-Wassermann subfactor) $\mathcal{N}_{pq}^m(I) \subset \mathcal{N}_{pq}^m(I^c)^\natural$*

Proof. $\mathfrak{Vir}_{1/2}(I)$ and $\mathfrak{Vir}_{1/2}(I^c)$ supercommute, then, by lemma 2.42 and Nelson's theorem, G_f and G_g supercommute for f and g concentrated on I and I^c. So is for the von Neumann algebra they generate. □

Theorem 2.54. *(Reeh-Schlieder theorem) Let $v \in H_{pq}^m$ be a non-null vector of finite level, then, $\mathcal{N}_{pq}^m(I).v$ is dense in H_{pq}^m (i.e. v is a cyclic vector).*

Proof. It's a general principle of local algebra, see [25] p 502. □

2.5 Real and complex fermions

Reminder 2.55. *(The complex Clifford algebra, see [25]) Let* H *be a complex Hilbert space, the complex Clifford algebra* $Cliff(H)$ *is the unital* $\star$*-algebra generated by a complex linear map* $f \mapsto a(f)$ $f \in H$ *satisfying:*

$$[a(f), a(g)]_+ = 0 \quad \text{and} \quad [a(f), a(g)^\star]_+ = (f, g)$$

The complex Clifford algebra as a natural irreducible representation π *on the fermionic Fock space* $\mathcal{F}(H) = \Lambda H = \bigoplus_{n=0}^{\infty} \Lambda^n H$ *(with* $\Lambda^0 H = \mathbb{C}\Omega$ *and* Ω *the vacuum vector), given by* $\pi(a(f))\omega = f \wedge \omega$ *bounded. Let* $c(f) = a(f) + a(f)^\star$ *satisfying* $[c(f), c(g)]_+ = 2Re(f, g)$ *and generating the real Clifford algebra. Warning,* c *is only* $\mathbb{R}$*-linear. We have the correspondence* $a(f) = \frac{1}{2}(c(f) - ic(if))$*. Now if* P *is a projector on* H*, we can define a new irreducible representation* π_P *of the complex Clifford algebra by* $\pi_P(a(f)) = \frac{1}{2}(c(f) - ic(\mathcal{I}f))$*, where* $\mathcal{I}$ *is the multiplication by* i *on* PH *and by* $-i$ *on* $(I-P)H$*, ie,* $\mathcal{I} = iP - i(I-P) = i(2P - I)$*. We know that* π_P *and* π_Q *are unitary equivalent if* $P - Q$ *is an Hilbert-Schmidt operator. Now, a unitary* $u \in U(H)$ *is implemented in* π_P *if* $\pi_P(a(u.f)) = U\pi_P(a(f))U^\star$ *with* U *unitary, unique up to a phase. But* $\pi_P(a(u.f)) = \pi_Q(a(f))$ *with* $Q = u^\star P u$*. Then,* u *is implemented in* π_P *if* $[P, u]$ *is Hilbert-Schmidt.*

Reminder 2.56. *More generally, taking a real Hilbert space* H*, we have the real Clifford algebra:* $[c(f), c(g)]_+ = 2(f, g)$*,* $f, g \in H$*. Then, we define a complex structure* $\mathcal{I}$ *with* $\mathcal{I}^2 = -Id$*. We obtain the complex Hilbert space* $H_{\mathcal{I}}$ *and then we can define the complex Clifford algebra by:* $A(f) = \frac{1}{2}(c(f) - ic(\mathcal{I}f))$*, acting irreducibly on the fermionic Fock space* $\mathcal{F}_{\mathcal{I}} = \Lambda H_{\mathcal{I}}$*. Now, the quantisation condition is:* $u \in O(H)$ *is implemented in* $\mathcal{F}_{\mathcal{I}}$ *if* $[u, \mathcal{I}]$ *is Hilbert-Schmidt. This quantisation due to Segal can be deduce form the condition on the complex case, using the doubling construction described below.*

Example 2.57. *(The Neveu-Schwarz real fermions)*
Let the real Hilbert space of anti-periodic functions $H_{NS} = \{f : \mathbb{R} \to \mathbb{R} | f(\theta + 2\pi) = -f(\theta)\}$ *with basis,* $\{cos(r\theta), sin(r\theta) | r \in \mathbb{Z} + 1/2\}$*, let the complex structure* $\mathcal{I}$ *defined by* $\mathcal{I}cos(r\theta) = sin(r\theta)$ *and* $\mathcal{I}sin(r\theta) = -cos(r\theta)$*. Then we obtain the operators* $c(f)$ *acting irreducibly on the fermionic Fock space we call* $\mathcal{F}_{NS}$*. Then, we define* $\psi_n = c(cos(n\theta)) + ic(sin(n\theta))$*. Now,* $\psi_n^\star = \psi_{-n}$ *and* $[\psi_m, \psi_n]_+ = \delta_{m+n} Id$*. The fermionic Fock space can be identified with the irreducible positive energy representation already studied.*

Reminder 2.58. *(The doubling construction) This is a precise mathematical version of the following physicists slogan: "a complex fermion is equivalent to two real fermions ". We start with a real Hilbert space H and we take $H \oplus iH$ (as a real Hilbert space, iH is the same as H). Let $v = \xi \oplus i\eta$, we define a real Clifford algebra by $c(v) = c(\xi) + c(i\eta)$, acting irreducibly on $\mathcal{F}(H) \otimes \mathcal{F}(iH)$. Then we define $a(v) = \frac{1}{2}(c(v) - ic(iv))$ satisfying the complex Clifford relation on the complex Hilbert space $H \oplus iH$. The operator $\mathcal{I}$ on H extends naturally into a unitary operator on $H \oplus iH$. Now, because $\mathcal{I}^2 = -Id$, it has the form $\mathcal{I} = i(2P - I)$, with P an orthogonal projection. Then the action of the operator $a(v)$ on $\mathcal{F}(H) \otimes \mathcal{F}(iH)$ can be identified with the representation π_P above, by the unique unitary sending $\Omega \otimes \Omega$ to Ω.*

Example 2.59. *We apply to the previous example: in this case, $H_{NS} \oplus iH_{NS} = \{f : \mathbb{R} \to \mathbb{C} | f(\theta + 2\pi) = -f(\theta)\}$. But then the multiplication with $e^{i\theta/2}$ gives an identification with $L^2(\mathbb{S}^1, \mathbb{C})$. This construction was already use on [19].*

Reminder 2.60. *(The local algebra for complex fermions) Let V be a complex finite dimensional complex vector space and $H = L^2(\mathbb{S}^1, V)$, let P be the projection on the Hardy space $H^2(\mathbb{S}^1, V)$ (the space of function without negative Fourier coefficient). Let I be a proper interval of $\mathbb{S}^1$ and $\mathcal{M}(I)$ be the von Neumann algebra generated by $\pi_P(Cliff(L^2(I, V)))$, then:*

(a) (Haag-Araki duality) $\quad \mathcal{M}(I)^\natural = \mathcal{M}(I^c)$

(b) (Covariance) $u_{\varphi^{-1}} : f \mapsto \sqrt{\varphi'}.f \circ \varphi$ defines a unitary action of $\varphi \in$Diff$(\mathbb{S}^1)$ on H; this action is implemented in π_P.

(c) The modular action on $\mathcal{M}(I)$ is $\sigma_t(x) = \pi_P(\varphi_t) x \pi_P(\varphi_t)^\star$, with $\varphi_t \in$ Diff$(\mathbb{S}^1)$ the Möbius flow fixing the end point of I. For example, if I is the upper half-circle, then $\partial I = \{-1, +1\}$ and $\varphi_t(z) = \frac{ch(t)z + sh(t)}{sh(t)z + ch(t)}$.

(d) The modular action is ergodic (ie it fixes only the scalar operators), so that $\mathcal{M}(I)$ is a III_1 factor (the hyperfinite one).

Remark 2.61. *By the doubling construction, Diff$(\mathbb{S}^1)$ acts on H_{NS} by:*

$$\pi(\varphi)^{-1}.f = |\varphi'|^{1/2} f \circ \varphi$$

and the action is quantised. We verify directly that $H_{\mathbb{C}} := H_{NS} \oplus iH_{NS}$ *admits the orthogonal basis* $e_r = e^{ir\theta}$ *with* $r \in \mathbb{Z} + 1/2$, *that* $\mathcal{I} = (2P - I)i$, *with* P *the Hardy projection (on the positive modes* $r \geq 0$*). Now, the Lie algebra of Diff*$(\mathbb{S}^1)$ *is the Witt algebra. The infinitesimal version of the previous action is* $d_n e_r = -(r + n/2)e_{r+n}$*: the action of the Witt algebra on the* $1/2$*-density (see below or [10] p 4). This infinitesimal action of the Witt algebra is implemented on the Fock space* $\mathcal{F}_{\mathbb{C}} = \mathcal{F}_{NS} \otimes \mathcal{F}_{NS}$ *into the Virasoro derivation on the real fermions:* $[L_n, \psi_r] = -(r + n/2)\psi_{r+n}$ *(consistent with section* **??***). Let* $SU(1,1)$ *be the group of* $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ *with* $|\alpha|^2 - |\beta|^2 = 1$. *By the Mobius transformation:* $g(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$, $SU(1,1)$ *is injected in Diff*$(\mathbb{S}^1)$, *and its Lie algebra is generated by* d_{-1}, d_0, d_1. *Now, we can see directly that* $SU(1,1)$ *is quantised, because it acts unitarily and commutes with* P*:*

$$\pi(g)^{-1} f(z) = \tfrac{1}{|\bar{\beta} z + \bar{\alpha}|} f(g(z))$$

Using $|\bar{\beta} z + \bar{\alpha}| = (\bar{\beta} z + \bar{\alpha})^{1/2} (\beta \bar{z} + \alpha)^{1/2}$, *for* $k \geq 0$*:*

$$\pi(g)^{-1} z^{k+1/2} = \tfrac{(\alpha z + \beta)^k z^{1/2}}{(\bar{\beta} z + \bar{\alpha})^{k+1}} \in PH_{\mathbb{C}}$$

Now, the quantised action of $SU(1,1)$ *fixes the vacuum vector of the fermionic Fock space, because* L_{-1}, L_0, L_1 *vanish on the vacuum vector.*
Note that the Lie algebra of the modular action is generated by $L_1 - L_{-1}$.

Reminder 2.62. *(Takesaki devissage [20]) Let* $M \subset B(H)$ *be a von Neumann algebra,* $\Omega \in H$ *cyclic for* M *and* M', $\triangle^{it}$, J *the corresponding modular operators (*$\triangle^{it} M \triangle^{-it} = M$ *and* $JMJ = M'$*). If* $N \subset M$ *is a von Neumann subalgebra such that* $\triangle^{it} N \triangle^{-it} = N$ *(conditional expectation), then:*

(a) $\triangle^{it}$ *and* J *restrict to the modular automorphism group* $\triangle_1^{it}$ *and conjugation operator* J_1 *of* N *for* Ω *on the closure* H_1 *of* $N\Omega$.

(b) $\triangle_1^{it} N \triangle_1^{-it} = N$ *and* $J_1 N J_1 = N'$ *on* H_1.

(c) If p *is the projection onto* H_1, *then* $pMp = Np$ *and*
$N = \{x \in M \mid xp = px\}$ *(the Jones relations [6])*

(d) $H_1 = H \iff M = N$

(e) *The modular group fixes the center. In fact* $\triangle^{it}x\triangle^{-it} = x$ *and* $JxJ = x^\star$ *for* $x \in Z(M) = M \cap M'$.

Definition 2.63. *Let* $\mathcal{M}_{NS}(I)$ *be the von Neumann algebra generated by the real Neveu-Schwarz* ψ_f *with* f *localised on* I.

Lemma 2.64. *(Reeh-Schlieder theorem) Let* $v \in \mathcal{F}_{NS}$ *be a non-null vector of finite level, then,* $\mathcal{M}_{NS}(I).v$ *is dense in* $\mathcal{F}_{NS}$ *(i.e.* v *is a cyclic vector).*

Proof. It's a general principle of local algebra, see [25]. □

Reminder 2.65. *A von Neumann algebra* $\mathcal{M}$ *is hyperfinite iff it is injective, ie* $\mathcal{M} \subset B(H)$ *with conditional expectation (see [1]).*

Proposition 2.66. *The local algebra* $\mathcal{M}_{NS}(I)$ *satisfy Haag-Araki duality, covariance for Diff*$(\mathbb{S}^1)$*, and the modular action is geometric and ergodic. In particular,* $\mathcal{M}_{NS}(I)$ *is the hyperfinite* III_1 *factor*

Proof. The covariance is shown in remark 2.61. Then, $\mathcal{M}_{NS}(I)$ is stable by the modular action of $\mathcal{M}(I)$. Now, $\pi_P(\mathcal{M}_{NS}(I)) \subset \mathcal{M}(I) \subset B(H_\mathbb{C})$ with conditional expectation, so $\pi_P(\mathcal{M}_{NS}(I))$ is hyperfinite. Next by Takesaki devissage the modular action of $\pi_P(\mathcal{M}_{NS}(I))$ is ergodic, so it's the hyperfinite III_1 factor. Now, by definition of the type III, every subrepresentations are equivalents, but one copy of $\mathcal{F}_{NS}$ is a subrepresentation. So $\mathcal{M}_{NS}(I)$ is the hyperfinite III_1 factor. Finally, the Haag-Araki duality for $\mathcal{M}_{NS}(I)$ comes from the Haag-Araki duality for $\mathcal{M}(I)$, the Reeh-Schlieder theorem and the Takesaki devissage. □

2.6 Properties of local algebras deducable by devissage from loop superalgebras

In this section we will deduce a few partial results on the local von Neumann algebra of Neveu-Schwarz, using devissage from the loop superalgebras, but it's not enough. In the next section, we will prove more general definitive result by devissage from real and complex fermions (in particular this will imply all the result proved here).

Remark 2.67. $\mathcal{F}_{NS}^{\mathfrak{g}} = \mathcal{F}_{NS}^{\otimes 3}$.

Lemma 2.68. *Let N_1, N_2 be von Neumann algebra, with modular action $\sigma_t^{\Omega_1}$ and $\sigma_t^{\Omega_2}$, then, the modular action on $N_1 \overline{\otimes} N_2$ is $\sigma_t^{\Omega_1 \otimes \Omega_2} = \sigma_t^{\Omega_1} \otimes \sigma_t^{\Omega_2}$*

Proof. By KMS uniqueness (see [25] p 493). □

Definition 2.69. *Let $L(j, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ be the irreducible representation of the $\mathfrak{g}$-supersymmetric algebra $\widehat{\mathfrak{g}}$. Let the local von Neumann algebra $\mathcal{N}_j^\ell(I)$ generated by $\pi_j^\ell(g) \otimes \pi_{NS}^{\mathfrak{g}}(g)$ and $1 \otimes x$, with $g \in L_I G$ and $x \in \mathcal{M}_{NS}^{\mathfrak{g}}(I)$.*

Proposition 2.70. $\mathcal{N}_j^\ell(I) = \pi_j^\ell(L_I G) \otimes \mathcal{M}_{NS}^{\mathfrak{g}}(I)$.

Proof. $\pi_{NS}^{\mathfrak{g}}(g)$ supercommutes with $\mathcal{M}_{NS}^{\mathfrak{g}}(I^c)$, so by the Haag-Araki duality $\pi_{NS}^{\mathfrak{g}}(g) \in \mathcal{M}_{NS}^{\mathfrak{g}}(I)$. We deduce that $\mathcal{N}_j^\ell(I)$ is generated by $\pi_j^\ell(g) \otimes 1$ and $1 \otimes x$. The result follows. □

Theorem 2.71. *Combining the work of A. Wassermann [25] on local loop group and the previous work on Neveu-Schwarz fermions, we obtain*

(a) *(Local equivalence) For every representations H_j^ℓ, there is a unique $\star$-isomorphism $\pi_j^\ell : \mathcal{N}_0^\ell(I) \to \mathcal{N}_j^\ell(I)$ coming from $\pi_0^\ell(B_f^a) \mapsto \pi_j^\ell(B_f^a) = U.\pi_0^\ell(B_f^a).U^\star$ and $\pi_0^\ell(\psi_g^a) \mapsto \pi_j^\ell(\psi_g^a) = U.\pi_0^\ell(\psi_g^a).U^\star$, with $U : H_0^\ell \to H_j^\ell$ unitary.*

(b) *(Covariance) $\varphi \in$Diff$(\mathbb{S}^1)$ acts unitarily on H_j^ℓ with $\pi_j^\ell(\varphi) B_f^a \pi_j^\ell(\varphi)^\star = B_{f \circ \varphi^{-1}}^a$ and $\pi_j^\ell(\varphi) \psi_g^b \pi_j^\ell(\varphi)^\star = \psi_{\alpha . g \circ \varphi^{-1}}^b$, with $\alpha = \sqrt{(\varphi^{-1})'}$, a kind of Radon-Nikodym correction (which preserves the group action) to be compatible with the Lie structure, ie be unitary on $L^2(\mathbb{S}^1)_{\mathbb{R}}$.*

(c) *The modular action on $\mathcal{N}_0^\ell(I)$ is $\sigma_t(x) = \pi_0^\ell(\varphi_t) x \pi_0^\ell(\varphi_t)^\star$, with $\varphi_t \in$ Diff$(\mathbb{S}^1)$ the Möbius flow fixing the end point of I. For example, if I is the upper half-circle, then $\partial I = \{-1, +1\}$ and $\varphi_t(z) = \frac{ch(t) z + sh(t)}{sh(t) z + ch(t)}$.*

(d) *$\mathcal{N}_j^\ell(I)$ is the hyperfinite III_1 factor.*

(e) *$\mathcal{N}_0^\ell(I) = \mathcal{N}_0^\ell(I^c)^\natural$ (Haag-Araki duality)*

(f) *$\mathcal{N}_j^\ell(I) \subset \mathcal{N}_j^\ell(I^c)^\natural$ (Jones-Wassermann subfactor)*

(g) *$\mathcal{N}_j^\ell(I)^\natural \cap \mathcal{N}_j^\ell(I^c)^\natural = \mathbb{C}$ (irreducibility of the subfactor)*

Lemma 2.72. *The operators G_f and L_h act continuously on $\mathcal{H}_j^\ell$, the L_0-smooth completion of $L(j,\ell)\otimes\mathcal{F}_{NS}^{\mathfrak{g}}$.*

Proof. $\mathcal{H}_j^\ell$ decompose into some irreducible smooth representations of the discrete series $(\mathcal{H}_{pq}^m)$, the result follows by corollary 2.39 $\square$

Notation 2.73. *Let $p=2j+1$, $q=2k+1$ and $m=\ell+2$, then, from now, we can note $\mathcal{H}_{pq}^m$ as $\mathcal{H}_{jk}^\ell$. It will be a more conveniant notation for the fusion rules computations*

Reminder 2.74. *(Kac-Todorov coset construction) ([14] section 2.2 or [9]).*

$$\mathcal{H}_0^0\otimes\mathcal{H}_j^\ell=\bigoplus\nolimits_{\substack{1\le q\le m+1\\ p\equiv q[2]}}\mathcal{H}_{jk}^\ell\otimes\mathcal{H}_k^{\ell+2},\ \textit{and}$$

$$\pi_0^0(G_f)\otimes I+I\otimes\pi_j^\ell(G_f)=\sum[\pi_{jk}^\ell(G_f)\otimes I+I\otimes\pi_k^{\ell+2}(G_f)]$$

Lemma 2.75. *We write some usefull relations on $\mathcal{H}_j^\ell$:*

(a) $[\psi_f^a,\psi_h^b]_+=\delta_{a,b}(f,h)_{\mathbb{R}}$

(b) $[B_f^a,B_h^b]=[B^a,B^b]_{f.h}+(\ell+2)\delta_{a,b}(d(f),h)_{\mathbb{R}}$

(c) $[G_f,B_h^a]=-(\ell+2)^{1/2}\psi_{f.d(h)}^a$

(d) $[G_f,\psi_h^a]_+=(\ell+2)^{-1/2}B_{f.h}^a$

Proof. Direct by computation from section 4 of [13]. $\square$

Let π be a positive energy representation of the loop superalgebra $\widehat{\mathfrak{g}}$. We know, it is always of the form $H\otimes\mathcal{F}_{NS}^{\mathfrak{g}}$, where H is a positive energy representation σ of LG (non necessarily irreducible). The Clifford algebra acts on the second factor and the loop group acts by tensor product. We have already seen that the von Neumann algebra $\pi(\widehat{\mathfrak{g}}_I)''$ is naturally a tensor product of von Neumann algebras (proposition 2.70). On the other hand, we have the operators $\pi(L_f)$, $\pi(G_f)$, given by the Sugawara construction (first sections) and $\pi(\varphi)$ with $\varphi\in$Diff($\mathbb{S}^1$). The L_f gives a projective representation of the Witt algebra, so exponentiate them give the element of Diff($\mathbb{S}^1$). The action of Diff($\mathbb{S}^1$) is also given by a tensor product of representation. The following property will be fundamental.

Theorem 2.76. *$\pi(\varphi) \in \pi(\widehat{\mathfrak{g}}_I)''$ and $\pi(L_f)$, $\pi(G_f)$ are affilated to $\pi(\widehat{\mathfrak{g}}_I)''$, if φ and f are concentrate on I^c.*

Remark 2.77. *We will prove it for Diff$(\mathbb{S}^1)$, and so for the L_f, in general, but for G_f, only for the vacuum representation (general proof on the next section).*

Proof. For the vacuum representation, it's an immediate consequence of the Haag-Araki duality. Now, we can restrict to π irreducible. For Diff$(\mathbb{S}^1)$, because we have Haag-Araki duality on $\mathcal{F}_{NS}^{\mathfrak{g}}$, it's sufficient to prove that $\sigma(\mathrm{Diff}_I(\mathbb{S}^1))'' \subset \sigma(L_I G)''$. By local equivalence, there exists a unitary U intertwining σ and the vacuum representation σ_0. By Haag duality and covariance $\sigma_0(\mathrm{Diff}_I(\mathbb{S}^1))'' \subset \sigma_0(L_I G)''$. Then, $U\sigma_0(\varphi)U^\star \subset \sigma(L_I G)'' \subset \sigma(L_{I^c}G)'$. On the other hand, $\sigma(\varphi) \in \sigma(L_{I^c}G)'$. So, $T = \sigma(\varphi^{-1})U\sigma_0(\varphi)U^\star \in \sigma(L_{I^c}G)'$. But, $T \in \sigma(L_I G)'$ by covariance relation. Now, by irreducibility $\sigma(L_I G)' \cap \sigma(L_{I^c}G)' = \mathbb{C}$, so T is a constant. The result follows. ☐

Theorem 2.78. *Haag-Araki duality holds for the Neveu-Schwarz algebra.*

Proof. Let K_0 be the vacuum representation Π_0 of $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$. The operators L_f and G_f of the coset construction act on K_0. We will prove that if f is concentrate on the interval I, G_f is affiliated with $\Pi_0(\widehat{\mathfrak{g}_I} \oplus \widehat{\mathfrak{g}_I})''$. Then, because $[G_{f_1}, G_{f_2}]_+ = L_{f_1 f_2} + \textit{constant}$, L_f is also affiliated. By Haag-Araki duality, it suffices to prove that the operators G_f supercommutes with the bosons (elememt of the loop algebra) and the fermions, concentrate on I^c. Let $A = G_f$ and let B be either the bosonic operator or the fermionic operator conjugate by the Klein transformation. They are formally self-adjoint for f real. By relation 2.75, they commute formally. By the Sobolev estimates and the Glimm-Jaffe-Nelson theorem, $A^2 + B^2$ is essentially self-adjoint. So Nelson's theorem imply the commutation in term of bounded function.

Now, by the coset construction, and the Reeh-Schlieder theorem, the bounded functions of the G_f and L_f applied on the vacuum vector of K_0 generate the vacuum positive energy representation of the Neveu-Schwarz algebra. The Haag-Araki duality follows by Takesaki devissage. ☐

Lemma 2.79. *(Covariance) Let $\varphi \in$ Diff$(\mathbb{S}^1)$, then $\pi_j^\ell(\varphi)\pi_j^\ell(G_f)\pi_j^\ell(\varphi)^\star = \pi_j^\ell(G_{\beta.f\circ\varphi^{-1}})$, with $\beta = 1/\alpha$, and $\alpha = \sqrt{(\varphi^{-1})'}$ and $f \in C^\infty(\mathbb{S}^1)$.*

Proof. $\pi_j^\ell(\varphi)[G_f, B_h^a]\pi_j^\ell(\varphi)^\star = -(\ell+2)^{-1/2}\psi^a_{\alpha.(f\circ\varphi^{-1}).(d(h)\circ\varphi^{-1})} =$
$-(\ell+2)^{-1/2}\psi^a_{\beta.(f\circ\varphi^{-1}).d(h\circ\varphi^{-1})} = [G_{\beta.f\circ\varphi^{-1}}, \pi_j^\ell(\varphi)B_h^a\pi_j^\ell(\varphi)^\star]$
Idem, $\pi_j^\ell(\varphi)[G_f, \psi_h^a]_+\pi_j^\ell(\varphi)^\star = [G_{\beta.f\circ\varphi^{-1}}, \pi_j^\ell(\varphi)\psi_h^a\pi_j^\ell(\varphi)^\star]_+$.
Then, by irreducibility, $\pi_j^\ell(\varphi)G_f\pi_j^\ell(\varphi)^\star - G_{\beta.f\circ\varphi^{-1}}$ is a constant operator; it's also an odd operator, so it's zero. □

Corollary 2.80. *By the coset construction, the covariance relation runs also on the discrete series representations of the Neveu-Schwarz algebra.*

2.7 Local algebras and fermions

In [25], the representation of $LSU(2)$ at level 1 are constructed using two complex fermions. This corresponds to the complex Clifford algebra construction on $\Lambda(L^2(\mathbb{S}^1, \mathbb{C}^2)) = \mathcal{F}_{\mathbb{C}^2}$. The level ℓ representations are obtained taking $\mathcal{F}_{\mathbb{C}^2}^{\otimes\ell}$. Then, the level ℓ representations of the corresponding loop superalgebra are realized on the tensor product of this Fock space and the space $\mathcal{F}_{NS}^{\mathfrak{g}}$, of three fermions. As vertex superalgebra, the vertex superalgebra of the loop superalgebra defines a vertex sub-superalgebra of the vertex superalgebra of $\mathcal{F}_{\mathbb{C}^2}^{\otimes\ell} \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$.

Let $H = \{f : \mathbb{R} \to \mathbb{R} \ | f(x+2\pi) = -f(x)\}$. Let $\mathcal{F}_{NS}^V = \Lambda(H \otimes V)$, then, $\mathcal{F}_{NS}^{V_1\oplus V_2} = \mathcal{F}_{NS}^{V_1} \otimes \mathcal{F}_{NS}^{V_2}$. Now, considering the diagonal inclusion $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$, $H \otimes (\mathfrak{g} \oplus \mathfrak{g}) \ominus H \otimes \mathfrak{g} = H \otimes [(\mathfrak{g} \oplus \mathfrak{g}) \ominus \mathfrak{g}] = H \otimes [(\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{g}]$. Then, we easily seen that in the Kac-Todorov construction described before:

$$\mathcal{F}_{NS}^{\mathfrak{g}} \otimes (\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(i,\ell)) = \bigoplus L(c_m, h_{pq}^m) \otimes (\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(j, \ell+2)),$$

we can simplify by a factor $\mathcal{F}_{NS}^{\mathfrak{g}}$ to obtain the following GKO construction:

$$\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(i,\ell) = \bigoplus L(c_m, h_{pq}^m) \otimes L(j, \ell+2)),$$

preserving the coset action of the Neveu-Schwarz algebra. It's also true replacing $L(i,\ell)$ by a (non necessarily irreducible) positive energy representation $\mathcal{H}$ of level ℓ. Then the coset action of the Neveu-Schwarz algebra on $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes \mathcal{H}$ is described by (see also [3] p114):

(a) $L_n^{gko} = L_n^{\mathfrak{g}\oplus\mathfrak{g}} - L_n^{\mathfrak{g}}$

(b) $G^{gko}(z) = \sum G_r^{gko} z^{-r-3/2} = \Phi(\tau_{gko}, z)$
with Φ the module-vertex operator on $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes \mathcal{H}$ (see [13] section 4.3), and $\tau_{gko} = (2(\ell+2)(\ell+4))^{-1/2}(\ell\tau_1 - 2\tau_2)$, with τ_1, τ_2 as in [13] definition 4.38 .

Note that: $\Phi(\ell\tau_1 - 2\tau_2, z) = [\sum_k (\ell\psi_k(z) \otimes X_k(z) - I \otimes \psi_k(z)S_k(z))]$
$= [\sum_k (\ell\psi_k(z) \otimes X_k(z) - \frac{i}{3}\sum_{ij} \Gamma_{ij}^k I \otimes \psi_i(z)\psi_j(z)\psi_k(z))]$
$= [\ell \sum_k (\psi_k(z) \otimes X_k(z) - 2i\sqrt{2} I \otimes \psi_1(z)\psi_2(z)\psi_3(z)]$.

Now, $\mathcal{F}_{\mathbb{C}^2}^{\otimes \ell}$ is a level ℓ representation of the loop algebra (containing all the irreducibles). We apply the previous GKO construction on $\mathcal{F}_{\mathbb{C}^2}^{\otimes \ell} \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$. Let $\mathcal{N}(I) = \mathcal{M}(I)^{\otimes \ell} \otimes \mathcal{M}_{NS}^{\mathfrak{g}}(I)$ be the local von Neumann algebra generated by the corresponding real and complex fermions. Let π_{gko} be the coset representation of $\mathfrak{Vir}_{1/2}$ on. Now, as previously, $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))$ supercommutes with $\mathcal{N}(I^c)$, then by Haag-Araki duality $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))'' \subset \mathcal{N}(I)$. Now, π_{gko} is a direct sum of all the irreducible positive energy representation π_i (with multiplicities) of the Neveu-Schwarz algebra. As previously (see lemma 2.79), $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))''$ is stable under the modular action of $\mathcal{N}(I)$. So we can apply the Takesaki devissage. We deduce that $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))''$ is the hyperfinite III_1 factor. By the property of the type III, every subrepresentations of π_{gko} are equivalents; in particular all the $\pi_i(\mathfrak{Vir}_{1/2}(I))''$ are the hyperfinite III_1-factor, and are equivalents to $\pi_0(\mathfrak{Vir}_{1/2}(I))''$: it's the local equivalence for the Neveu-Schwarz algebra. Finally, let Ω be the vacuum vector of $\mathcal{F}_{\mathbb{C}^2}^{\otimes \ell} \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$, then clearly $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))''\Omega$ is dense (Reeh-Schlieder theorem) on the vacuum representation of $\mathfrak{Vir}_{1/2}$ tensor its corresponding multiplicity M_0. Let P be the projection on, then P commutes with the modular operators (because the vacuum vector is invariant) and with the Klein operator κ. But by Takeaki devissage $PN(I)P = \pi_{gko}(\mathfrak{Vir}_{1/2}(I))''P = [\pi_0^{\otimes M_0}(\mathfrak{Vir}_{1/2}(I))'']$.
So $\kappa JPN(I)PJ\kappa^\star = P\kappa JN(I)J\kappa^\star P = PN(I)^\natural P = PN(I^c)P$
$= [\pi_0^{\otimes M_0}(\mathfrak{Vir}_{1/2}(I^c))''] = [\pi_0^{\otimes M_0}(\mathfrak{Vir}_{1/2}(I))^\natural]$. The Haag-Araki duality for the Neveu-Schwarz algebra follows.

Corollary 2.81. *(Generalized Haag-Araki duality)*
$\pi_{gko}(\mathfrak{Vir}_{1/2}(I))'' = \pi_{gko}(\mathfrak{Vir}_{1/2})'' \cap \mathcal{N}(I)$

Corollary 2.82. $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ *is generated by chains of compressed fermions concentrate in* I.

Proof. Immediate from Jones relation: $p_0\mathcal{N}(I)p_0 = \pi_{gko}(\mathfrak{Vir}_{1/2}(I))''p_0$. □

Now because π_{gko} contains all the irreducible positive energy representations π_i of charge c_m , we deduce that:

Corollary 2.83. *Let π be the direct sum of all the π_i.*
To simplify we note $\pi = \pi_0 \oplus ... \oplus \pi_n$. Then $\mathcal{A} := \pi(\mathfrak{Vir}_{1/2}(I))''$
$= \{T = \begin{pmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_n \end{pmatrix} \mid T$ *supercommutes with* $\mathcal{B}\}$ *with* $\mathcal{B} = \{\begin{pmatrix} S_{11} & \dots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & ... & S_{nn} \end{pmatrix}$
such that S_{ij} is a chain of compressed fermions $p_i\phi(f)p_j$ concentrate on I^c $\}$

By definition $\mathcal{A}^\natural = \mathcal{B}$. Now, let q_i the projection on π_i, then $q_i \in \mathcal{A}^\natural$, so, $(q_i\mathcal{A})^\natural = p_i\mathcal{B}p_i$. Then $(q_i\mathcal{A})^\natural = \pi_i(\mathfrak{Vir}_{1/2}(I))^\natural = \{S_{ii} \mid ...\}$.

Corollary 2.84. *$\pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural$ is generated by chains of compressed fermions concentrated in I.*

Remark 2.85. *In the next section, we will see by unicity that the compression of complex fermions give a primary fields of charge $\alpha = (1/2, 1/2)$, and the compression of a real fermions give primary fields of charge $\beta = (0, 1)$.*

Remark 2.86. *We will see that the supercommutation relation on the vacuum (Haag-Araki duality) is replaced by braiding relations of primary fields. As consequence, we directly see that $\pi_i(\mathfrak{Vir}_{1/2}(I))^\natural$ and $\pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural$ do not necessarily supercommute if $i \neq 0$. Then, the formulation of the local von Neumann algebra, generated by chains of primary fields (with braiding), shows explicitly the failure of Haag-Araki duality outside of the vacuum.*

3 Primary fields

3.1 Primary fields for $LSU(2)$

This section is an overview of the primary field theory of $LSU(2)$, for a more detailed exposition see [25] and [21]. It would be convenient to also cite [15] and [2]. Let V be a representation of $G = SU(2)$ or $\mathfrak{g} = \mathfrak{sl}_2$.

Definition 3.1. *Let λ, $\mu \in \mathbb{C}$, we define the ordinary representations of $L\mathfrak{g} \rtimes \mathfrak{Vir}$ as $\mathcal{V}_{\lambda,\mu}$, generated by (v_i), $v \in V$ and $i \in \mathbb{Z}$, and:*

(a) $L_n.v_i = -(i + \mu + n\lambda)v_{i+n}$

(b) $X_m.v_i = (X.v)_{m+i} \quad (X \in \mathfrak{g})$

Definition 3.2. *Let L_i^ℓ and L_j^ℓ be irreducible representation of $L\mathfrak{g}$, of level ℓ and spin i and j. We define a primary field as a linear operator:*

$$\phi : L_j^\ell \otimes \mathcal{V}_{\lambda,\mu} \to L_i^\ell$$

that intertwines the action of $L\mathfrak{g} \rtimes \mathfrak{Vir}$. We call V the charge of ϕ.

Reminder 3.3. *Let $h_i^\ell = \frac{i^2+i}{\ell+2}$ the lowest eigenvalue of L_0 on L_i^ℓ (see theorem* **??***). The eigenspace is the $\mathfrak{sl}_2$-module V_i.*

Definition 3.4. *For $w \in \mathcal{V}_{\lambda,\mu}$, let $\phi(w) : L_j^\ell \to L_i^\ell$*

Lemma 3.5. *Let $X \in L\mathfrak{g} \rtimes \mathfrak{Vir}$, then $[X, \phi(w)] = \phi(X.v)$*

Proof. As for the proof of lemma 3.31. □

Lemma 3.6. *ϕ non-null implies that $\mu = h_j^\ell - h_i^\ell$.*

Lemma 3.7. *(Gradation) $\phi(v_n).(L_j^\ell)_{s+h_j^\ell} \subset (L_i^\ell)_{s-n+h_i^\ell}$*

Definition 3.8. *Let $h = 1 - \lambda$ be the conformal dimension of ϕ, and $\triangle = 1 - \lambda + \mu = h + h_j^\ell - h_i^\ell$; we define:*

$$\phi(v, z) = \textstyle\sum_{n\in\mathbb{Z}} \phi(v_n) z^{-n-\triangle} \qquad (v \in V).$$

Lemma 3.9. *(Compatibility condition)*

(a) $[L_n, \phi(v,z)] = z^n[z\frac{d}{dz} + (n+1)h]\phi(v,z)$

(b) $[X_m, \phi(v,z)] = z^m\phi(X.v, z)$

Proof. Direct from the definition. ☐

Lemma 3.10. *If $\widetilde{\phi}(z,v)$ satisfy the compatibility condition, then, it gives a primary fields for $LSU(2)$.*

Proof. It's an easy verification. ☐

Proposition 3.11. *(Initial term) A primary field $\phi : L_j^\ell \otimes \mathcal{V}_{\lambda,\mu} \to L_i^\ell$ with every parameters fixed, is completly determined by its initial term:*

$$\phi_0 : V_j \otimes V \to V_i$$

Proof. Idem, by intertwining relation; see [25] p 513 for details. ☐

Proposition 3.12. *(Unicity) If $V = V_k$ is irreducible, the space of such primary field is at most one-dimensional.*

Proof. ϕ_0 is an intertwining operator, ie, $\phi_0 \in Hom_{\mathfrak{g}}(V_j \otimes V_k, V_i)$ the multiplicity space at V_i of $V_j \otimes V_k = V_{|j-k|} \oplus V_{|j-k|+1} \oplus ... \oplus V_{j+k}$ (Clebsch-Gordan), so at most one-dimensional. ☐

Remark 3.13. *As for $\mathfrak{Vir}_{1/2}$ (see remark 3.39), with $(A_nB)(z)$ formula, we define inductively the $L\mathfrak{g}$-module L_k^ℓ from ϕ.*

Corollary 3.14. $\mu = h_j^\ell - h_i^\ell$ *and* $1 - \lambda = h = h_k^\ell$.

Definition 3.15. *We note ϕ as $\phi_{ij}^{k\ell}$, $\triangle$ as $\triangle_{ij}^{k\ell} = h_j^\ell - h_i^\ell + h_k^\ell$.*

We call ϕ a primary field of spin k; in our work, we just need to consider primary fields of spin $1/2$ and 1:

Proposition 3.16. *Up to a multiplication by a rational power of z:*
(a) The compression of complex fermions gives primary fields of spin $1/2$.
(b) The compression of real fermions gives primary fields of spin 1.

Proof. We just check the compatibility condition. The calculation can also be made on the vertex algebra of the fermions. See also [25] p 515. ☐

Definition 3.17. *We note $\phi_{ij}^{k\ell}$ be the primary field from L_j^ℓ to L_i^ℓ, of spin k. It's defined up to a multiplicative constant and is possibly null.*

Reminder 3.18. *(Constructible primary fields of spin $1/2$ or 1, see [25]).*

(a) $\phi_{ij}^{\frac{1}{2}\ell}$ *is non-null iff* $j = i \pm 1/2$ *and* $i + j + 1/2 \leq \ell$

(b) $\phi_{ij}^{1\ell}$ *is non-null iff* $j = i - 1, i,$ *or* $i + 1$ *and* $i + j + 1 \leq \ell$

with the restriction that: $0 \leq \ i, j \ \leq \ell/2$

Proposition 3.19. *Every primary fields $\phi_{ij}^{k\ell}(w) : L_j^\ell \to L_i^\ell$ of spin $k = 1/2$ or 1, are constructibles as compressions of complex and real fermions. respectively.*

Proof. For spin $1/2$ primary fields see [25] p 515.

Now, for spin 1: note that at level $\ell = 2$, there are only 0, $1/2$ and 1 as possible spins. But, the real Neveu-Schwarz fermions $\mathcal{F}_{NS}^{\mathfrak{g}}$ equals to $L_0^2 \oplus L_1^2$, and the real Ramond fermions $\mathcal{F}_R^{\mathfrak{g}}$ equals to $L_{1/2}^2$, as $LSU(2)$ module (see [14] corollary 5.7 and [3] p116). Then, compressions of the fermion field $\psi(z, v)$, with $v \in V_1 = \mathfrak{g}$ on $\mathcal{F}_{NS}^{\mathfrak{g}}$ or $\mathcal{F}_R^{\mathfrak{g}}$ give the spin 1 primary fields at level 2, by unicity and compatibility condition.

Now, $L_j^\ell \otimes L_k^{\ell'} = L_{|j-k|}^{\ell+\ell'} \oplus L_{|j-k|+1}^{\ell+\ell'} \oplus ... \oplus L_{j+k}^{\ell+\ell'}$, so:

(a) $\phi_{i,i-1}^{1\ell+2}(v)$ is the compression of $\phi_{01}^{1,2}(v) \otimes I : L_1^2 \otimes L_{i-1}^\ell \to L_0^2 \otimes L_{i-1}^\ell$.

(b) $\phi_{i,i+1}^{1\ell+2}(v)$ is the compression of $\phi_{10}^{1,2}(v) \otimes I : L_0^2 \otimes L_i^\ell \to L_1^2 \otimes L_i^\ell$.

(c) $\phi_{i,i}^{1\ell+2}(v)$ is the compression of $\phi_{01}^{1,2}(v) \otimes I : L_0^2 \otimes L_i^\ell \to L_1^2 \otimes L_i^\ell$.

The result follows. ☐

Corollary 3.20. *The primary fields of spin $k = 1/2$ or 1 are bounded and identifying the L^2-completion of $\mathcal{V}_{\lambda,\mu}$ with $L^2(\mathbb{S}^1, V_k)$, we obtain $\phi(f)$ for $f \in L^2(\mathbb{S}^1, V_k)$, with: $\|\phi(f)\| \leq K\|f\|_2$.*

Reminder 3.21. *(Braiding relations)*
In [21] and [25], the braiding relations of spin $1/2$ primary fields are given by reduced 4-point functions $f : \mathbb{C} \to W$, with W finite dimensional. We give an overview of this theory:

Let $F_j(z, w) = (\phi_{kj}^{\frac{1}{2}\ell}(u, z).\phi_{ji}^{\frac{1}{2}\ell}(v, w)\Omega_i, \Omega_k)$, *then by gradation it equals:*

$$\sum_{m\geq 0}(\phi_{kj}^{\frac{1}{2}\ell}(u,m).\phi_{ji}^{\frac{1}{2}\ell}(v,-m)\Omega_i,\Omega_k)z^{-m-\triangle}w^{m-\triangle'} = f_j(\zeta)z^{-\triangle}w^{-\triangle'}$$

with $f_j(\zeta) = \sum_{m\geq 0}(\phi_{kj}^{\frac{1}{2}\ell}(u,m).\phi_{ji}^{\frac{1}{2}\ell}(v,-m)\Omega_i,\Omega_k)\zeta^m$ *and* $\zeta = w/z$. *The function* f_j *are holomorphic for* $|\zeta| < 1$. *Now,* $\phi_{kj}^{\frac{1}{2}\ell}$ *and* $\phi_{ji}^{\frac{1}{2}\ell}$ *are non-zero field, ie* $Hom_{\mathfrak{g}}(V_j\otimes V_{1/2}, V_k)$ *and* $Hom_{\mathfrak{g}}(V_i\otimes V_{1/2}, V_j)$ *are* 1*-dimensional space. Then, the set of possible such* j *generate the space* $W = Hom_{\mathfrak{g}}(V_{1/2}\otimes V_{1/2}\otimes V_i, V_j)$. *Then, we consider the vector* $f_j(\zeta)$ *as a vector in* W. *Let* $\tilde{f}_j = \zeta^{\lambda_j}f_j$, *(with* $\lambda_j = (j^2+j-i^2-i-3/4)/(\ell+2)$*), called the reduced four points functions.* $\tilde{f}_j(z)$ *is defined on* $\{z : |z| < 1, z\notin [0,1[\}$. *It satisfy the Knizhnik-Zamolodchikov ordinary differential equation, equivalent to the hypergeometric equation of Gauss:*

$$\tilde{f}'(z) = A(z)\tilde{f}(z), \text{ with } A(z) = \frac{P}{z} + \frac{Q}{1-z}$$

with P, $Q \in End(W)$. *It's proved in [25] section 19, the existence of a holomorphic gauge transformation* $g : \mathbb{C}\backslash[1,\infty[\to GL(W)$ *with* $g(0) = I$ *such that:* $g^{-1}Ag - g^{-1}g' = P/z$. *The solution of the ODE is then* $\tilde{f}(z) = g(z)z^PT$, *with* T *an eigenvector of* P. *So, up to a power of* z, *the solutions* $\tilde{f}_j(z)$ *are just the columns of* $g(z)$ *(in the spectral base of* P*). Now, let* $r_j(z) = \tilde{f}_j(z^{-1})$ *on* $\{z : |z| > 1, z\notin [1,\infty[\}$, *then* r_j *satisfy clearly the equation:*

$$r'(z) = B(z)r(z), \text{ with } B(z) = \frac{Q-P}{z} + \frac{Q}{1-z}$$

The function r_j *and* $\tilde{f}_j$ *extend to holomorphic functions on* $\mathbb{C}\backslash[0,\infty[$. *It's proved in [25], that the solutions of these two equations are related by a transport matrix* $c = (c_{ij})$ *with* $c_{ij}\neq 0$, *so that:*

$$\tilde{f}_j(z) = \sum c_{jm}\tilde{f}_m(z^{-1})$$

We then obtain, up to an analytic continuation, the following equality:

$$(\phi_{kj}^{\frac{1}{2}\ell}(u,z).\phi_{ji}^{\frac{1}{2}\ell}(v,w)\Omega_i,\Omega_k) = \sum c_{jm}(\phi_{km}^{\frac{1}{2}\ell}(v,w).\phi_{mi}^{\frac{1}{2}\ell}(u,z)\Omega_i,\Omega_k)$$

This relation extends to any finite energy vectors:

$$(\phi_{kj}^{\frac{1}{2}\ell}(u,z).\phi_{ji}^{\frac{1}{2}\ell}(v,w)\eta,\xi) = \sum c_{jm}(\phi_{km}^{\frac{1}{2}\ell}(v,w).\phi_{mi}^{\frac{1}{2}\ell}(u,z)\eta,\xi)$$

This analysis runs idem for braiding relations between spin $1/2$ *and spin* 1 *primary fields, then:*

Theorem 3.22. *(Braiding relations)*
Let $(k_1, k_2) = (1/2, 1/2)$, $(1, 1/2)$ or $(1/2, 1)$; $v_1 \in V_{k_1}$ and $v_2 \in V_{k_2}$.

$$\phi_{ij}^{k_1\ell}(v_1, z)\phi_{jk}^{k_2\ell}(v_2, w) = \sum \mu_r \phi_{ir}^{k_2\ell}(v_2, w)\phi_{rk}^{k_1\ell}(v_1, z) \text{ with } \mu_r \neq 0$$

To simplify, we don't write the dependence of μ_r on the other coefficients.

Remark 3.23. *The way to write the braiding relations is a simplification. In fact, the left side is defined for $|z| < |w|$, and the right side for $|z| > |w|$, but each sides admit the same rational extension out of $z = w$. The braiding relations generalise the locality of vertex operator (see [13] definition 3.19).*

Remark 3.24. *To prove that all the coefficients are non-null for $(k_1, k_2) = (1, 1)$, we should solve Dotsenko-Fateev equations (see [19]).*

Reminder 3.25. *(Localised braiding relation) Let $f \in L^2(I, V_{k_1})$ and $g \in L^2(J, V_{k_2})$, with I, J be two disjoint proper intervals of $\mathbb{S}^1$. Using an argument of convolution (as [25] p 516), we can write the following localised braiding relations:*

$$\phi_{ij}^{k_1\ell}(f)\phi_{jk}^{k_2\ell}(g) = \sum \mu_r \phi_{ir}^{k_2\ell}(e_\alpha g)\phi_{rk}^{k_1\ell}(e_{-\alpha} f) \text{ with } \mu_r \neq 0$$

with $e_\alpha = e^{i\alpha\theta}$, $\alpha = h_i^\ell + h_k^\ell - h_j^\ell - h_r^\ell$ and (k_1, k_2) as previously.

Reminder 3.26. *(Contragrediant braiding) Let the previous ODE:*

$$\tilde{f}'(z) = A(z)\tilde{f}(z), \text{ with } A(z) = \frac{P}{z} + \frac{Q}{1-z}$$

and the previous gauge relation: $g^{-1}Ag - g^{-1}g' = P/z$.
In the same way, we can choose $h(z)$ with $h(0) = I$ and $hAh^{-1} - h'h^{-1} = -P/z$. This corresponds to take $-A(z)^t$ instead of $A(z)$. But then $(hg)' = [P, hg]/z$, which admits only the constant solutions, but $h(0)g(0) = I$, so $h(z) = g(z)^{-1}$. Then, the columns of $(g(z)^{-1})^t$ are the fundamental solutions of $k'(z) = -A(z)^t k(z)$. The transport matrix of this equation is just the transposed of the inverse of the original one, ie $(c^{-1})^t$.

3.2 Primary fields for $\mathfrak{Vir}_{1/2}$

Definition 3.27. *Let λ, $\mu \in \mathbb{C}$, $\sigma = 0, 1$, we define the ordinary representations of $\mathfrak{Vir}_{1/2}$ as $\mathcal{F}^{\sigma}_{\lambda,\mu}$, with base $(v_i)_{i\in\mathbb{Z}+\frac{\sigma}{2}}$, $(w_j)_{j\in\mathbb{Z}+\frac{1-\sigma}{2}}$, and:*

(a) $L_n.v_i = -(i + \mu + \lambda n)v_{i+n}$

(b) $G_s.v_i = w_{i+s}$

(c) $L_n.w_j = -(j + \mu + (\lambda - \frac{1}{2})n)w_{j+n}$

(d) $G_s.w_j = -(j + \mu + (2\lambda - 1)s)v_{j+s}$

Remark 3.28. *Let the space of densities $\{f(\theta)e^{i\mu\theta}(d\theta)^{\lambda} | f \in C^{\infty}(\mathbb{S}^1)\}$ where a finite covering of Diff$(\mathbb{S}^1)$ acts by reparametrisation $\theta \to \rho^{-1}(\theta)$ (if $\mu \in \mathbb{Q}$). Then its Lie algebra acts on too, so that it's a $\mathfrak{Vir}$-module vanishing the center (see [10]). Finally, an equivalent construction with superdensities gives a model for $\mathcal{F}^{\sigma}_{\lambda,\mu}$ as $\mathfrak{Vir}_{1/2}$-module (see [5]).*

Definition 3.29. *Let L^m_{pq} and $L^m_{p'q'}$ on the unitary discrete series of $\mathfrak{Vir}_{1/2}$. We define a primary field as a linear operator:*

$$\phi : L^m_{p'q'} \otimes \mathcal{F}^{\sigma}_{\lambda,\mu} \to L^m_{pq}$$

that superintertwines the action of $\mathfrak{Vir}_{1/2}$.

Definition 3.30. *For $v \in \mathcal{F}^{\sigma}_{\lambda,\mu}$, let $\phi(v) : L^m_{p'q'} \to L^m_{pq}$*

Lemma 3.31. *Let $X \in \mathfrak{Vir}_{1/2}$, then $[X, \phi(v)]_{\tau} = \phi(X.v)$*

Proof. We can suppose X to be homogeneous for the $\mathbb{Z}_2$-gradation τ. Now, ϕ superintertwines the action of $\mathfrak{Vir}_{1/2}$: $\phi.[X \otimes I + I \otimes X] = (-1)^{\partial X} X.\phi$
Let $\xi \otimes v \in L^m_{p'q'} \otimes \mathcal{F}^{\sigma}_{\lambda,\mu}$, then $\phi.[X \otimes I + I \otimes X](\xi \otimes v) = [\phi(v)X + \phi(Xv)]\xi$ and $X.\phi(\xi \otimes v) = X\phi(v)\xi$, then $[X, \phi(v)]_{\tau} = \phi(X.v)$. ☐

Lemma 3.32. *ϕ non-null implies that $\mu = h^m_{p'q'} - h^m_{pq}$.*

Lemma 3.33. *(Gradation)*

(a) $\phi(v_n).(L^m_{p'q'})_{s+h^m_{p'q'}} \subset (L^m_{pq})_{s-n+h^m_{pq}}$

(b) $\phi(w_r).(L^m_{p'q'})_{s+h^m_{p'q'}} \subset (L^m_{pq})_{s-r+h^m_{pq}}$

Definition 3.34. *Let $h = 1 - \lambda$ be the conformal dimension of ϕ, and $\triangle = 1 - \lambda + \mu = h + h^m_{p'q'} - h^m_{pq}$; we define:*

$$\phi(z) = \sum_{n \in \mathbb{Z} + \frac{\sigma}{2}} \phi(v_n) z^{-n-\triangle} \text{ and } \quad \theta(z) = \sum_{n \in \mathbb{Z} + \frac{1-\sigma}{2}} \phi(w_n) z^{-n-1/2-\triangle}$$

$\phi(z)$ is called the ordinary part and $\theta(z) = [G_{-1/2}, \phi(z)]$, the super part of the primary field.

Lemma 3.35. *(Covariance relations).*

(a) $[L_n, \phi(z)] = [z^{n+1} \frac{d}{dz} + h(n+1)z^n]\phi(z)$

(b) $[G_{n-1/2}, \phi(z)] = z^n \theta(z)$

(c) $[L_n, \theta(z)] = [z^{n+1} \frac{d}{dz} + (h + 1/2)(n+1)z^n]\theta(z)$

(d) $[G_{n-1/2}, \theta(z)]_+ = [z^n \frac{d}{dz} + 2hn.z^{n-1}]\phi_(z)$

Proof. Direct from the definition. □

Lemma 3.36. *(Compatibility condition)*

(a) $[L_n, \phi(z)] = [z^{n+1} \frac{d}{dz} + (n+1)z^n(1-\lambda)]\phi(z)$

(b) $[G_r, \phi(z)] = z^{r+1/2}[G_{-1/2}, \phi(z)]$

Proof. Immediate. □

Lemma 3.37. *If $\widetilde{\phi}(z)$ satisfy the compatibility condition, then, it gives a primary fields for the Neveu-Schwarz algebra, with $\widetilde{\theta}(z) = [G_{-1/2}, \widetilde{\phi}(z)]$ as super part.*

Proof. It's an easy verification. □

Proposition 3.38. *(Initial term) The space of primary fields $\phi : L^m_{p'q'} \otimes \mathcal{F}^\sigma_{\lambda,\mu} \to L^m_{pq}$ with every parameter fixed, is at most one-dimensional.*

Proof. Let Ω and Ω' be the cyclic vectors of the positive energy representations and $v \in \mathcal{F}^\sigma_{\lambda,\mu}$. Then, by the intertwining relations, $(\phi(v)\eta, \xi)$ is completly determined by the initial term $(\phi(v)\Omega, \Omega')$. Next $(\phi(v)\Omega, \Omega')$ is nonzero for v in a subspace of $\mathcal{F}^\sigma_{\lambda,\mu}$ of at most dimension one (lemma 3.33). □

Remark 3.39. *Using a slightly modified $(A_nB)(z)$ formula (see proposition **??**), we can inductively generate many fields from a given field ψ.*
For example we find:

$$(L_n\psi)(z) = [\sum C^r_{n+1}(-z)^r L_{n-r}]\psi(z) - \psi(z)[\sum C^r_{n+1}(-z)^{n+1-r} L_{r-1}]$$

We can also write a formula for G_r. Now, we see that:

$$(L_0\phi)(z) = [L_0, \phi(z)] - z[L_{-1}, \phi(z)] = h\phi(z)$$

*It's easy to see that using this machinery from $\phi(z)$ we generate the unitary $\mathfrak{Vir}_{1/2}$-module $L(h, c_m)$. Then, by FQS criterion, $h = h^m_{p''q''}$. We note the elements $\Phi(a, z)$ with $a \in L^m_{p''q''}$, $\phi(z) = \Phi(\Omega^m_{p''q''}, z)$ and if $\psi(z) = \Phi(a, z)$ then $(L_n\psi)(z) = \Phi(L_n.a, z)$. We do the same with G_r. We call Φ a general vertex operator, it generalizes the vertex operator of the section **??** , it admits many properties, but we don't need to enter into details.*

Corollary 3.40. $\mu = h^m_{p'q'} - h^m_{pq}$ *and* $1 - \lambda = h = h^m_{p''q''}$.

Definition 3.41. *We note ϕ as $\phi^{p''q''m}_{pqp'q'}$, $\triangle$ as $\triangle^{p''q''m}_{pqp'q'} = h^m_{p''q''} - h^m_{p'q'} + h^m_{pq}$.*

Definition 3.42. *With $p'' = 2k + 1$ and $q'' = 2k' + 1$, we call ϕ a primary field of charge (k, k').*

Note that the charge and the spaces between which the field acts fixes λ and μ, but σ can be 0 or 1. Now, $\sigma = 0$ or 1 corresponds to $\phi(z)$ with integers or half-integers modes respectively. On our work, we only need to consider primary fields of charge $\alpha = (1/2, 1/2)$ and $\beta = (0, 1)$:

Proposition 3.43. *Up to a multiplication by a rational power of z:*
(a) The compression of complex fermions gives primary fields of charge α.
(b) The compression of real fermions gives primary fields of charge β.

Proof. We just check the compatibility condition using the explicit formula of GKO for G_r. The calculation can also be made on the vertex algebra of the fermions. ∎

3.3 Constructible primary fields and braiding for $\mathfrak{Vir}_{1/2}$

Lemma 3.44. *Let $m = \ell + 2$. and* $\left\{ \begin{array}{lll} p = 2i+1 & p' = 2j+1 & p'' = 2k+1 \\ q = 2i'+1 & q' = 2j'+1 & q'' = 2k'+1 \end{array} \right.$

(a) $h_i^{\ell} = h_{pq}^{m} + h_{i'}^{\ell+2} - \frac{1}{2}(i-i')^2$

(b) $\triangle_{ij}^{k\ell} = \triangle_{pqp'q'}^{p''q''m} + \triangle_{i'j'}^{k'\ell+2} - C_{ii'jj'}^{kk'}$

with $C_{ii'jj'}^{kk'} = \frac{1}{2}[(i-i')^2 - (j-j')^2 + (k-k')^2]$

Proof. $h_{pq}^{m} = \frac{[(m+2)p-mq]^2-4}{8m(m+2)} = \frac{2p^2(m+2)-2q^2m-4}{8m(m+2)} + \frac{(p-q)^2}{8} = h_i^{\ell} - h_{i'}^{\ell+2} + \frac{1}{2}(i-i')^2$

Next, (b) is immediate by (a). $\square$

Notation 3.45. *We note $h_{ii'}^{\ell}$, $L_{ii'}^{\ell}$, $\phi_{ii'jj'}^{kk'\ell}$ and $\triangle_{ii'jj'}^{kk'\ell}$ instead of h_{pq}^{m}, L_{pq}^{m}, $\phi_{pqp'q'}^{p''q''m}$ and $\triangle_{pqp'q'}^{p''q''m}$.*

Definition 3.46. *A non-zero primary field of charge $\alpha = (1/2, 1/2)$ or $\beta = (0, 1)$ is called constructible if it's a compression fermions.*

Theorem 3.47. *(Constructible primary fields)*
(1)$\phi_{ii'jj'}^{\alpha\ell}$ is constructible iff:
$i + i' + 1/2 \leq \ell$ and $j + j' + 1/2 \leq \ell + 2$
(a) If $\sigma = 0$: $i' = i \pm 1/2$ and $j' = j \pm 1/2$,
(b) If $\sigma = 1$: $i' = i \pm 1/2$ and $j' = j \mp 1/2$.

(2) $\phi_{ii'jj'}^{\beta\ell}$ is constructible iff:
$i + i' \leq \ell$ and $j + j' + 1 \leq \ell + 2$
(a) If $\sigma = 0$: $i' = i$ and $j' = j \pm 1$.
(b) If $\sigma = 1$: $i' = i$ and $j' = j$

with the restriction that $0 \leq i, i' \leq \ell/2$ and $0 \leq j, j' \leq (\ell+2)/2$.

This section is devoted to prove the theorem.

Remark 3.48. *If we ignore σ, we see that the dimension of the spaces of constructible primary fields are 0, 1 or 2-dimensional, and it's correspond to the fusion rules obtained below.*

Remark 3.49. *The dimension of the space of all the primary fields (non necessarily construcible as above) have been calculated by Iohara and Koga [5], using the action on the singular vectors of $\mathcal{F}^{\sigma}_{\lambda,\mu}$. Their result shows that in the previous cases, every primary fields are constructibles.*

Corollary 3.50. *Let ϕ of charge α or β, then $\phi \neq 0$ iff ϕ constructible.*

Reminder 3.51. *(GKO construction, see [14] section 5)*

$$\mathcal{F}^{\mathfrak{g}}_{NS} \otimes L^{\ell}_{i} = \bigoplus L^{\ell}_{ij} \otimes L^{\ell+2}_{j}$$

and $\mathcal{F}^{\mathfrak{g}}_{NS} = L^2_0 \oplus L^2_1$ as $L\mathfrak{g}$-module.

Corollary 3.52. *(Braiding relations)*
Let $(\gamma_1, \gamma_2) = (\alpha, \alpha)$, (β, α) or (α, β):

$$\phi^{\gamma_1 \ell}_{ii'jj'}(z)\phi^{\gamma_2 \ell}_{jj'kk'}(w) = \sum \mu_{rr'}\phi^{\gamma_2 \ell}_{ii'rr'}(w)\phi^{\gamma_1 \ell}_{rr'kk'}(z) \text{ with } \mu_{rr'} \neq 0.$$

To simplify, we don't write the dependence of $\mu_{rr'}$ on the other coefficients.

proof of theorem 3.47 and corollary 3.52
This proof is an adaptation of the proof of Loke [11] for $\mathfrak{Vir}$.
I thank A. Wassermann to have simplified it.
Let H^{ℓ}_{j}, $H^{\ell}_{jj'}$ be the L^2-completion of L^{ℓ}_{j} and $L^{\ell}_{jj'}$.
Let $\Phi(v, z) = I \otimes \phi^{\frac{1}{2}\ell}_{ij}(v, z) : \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{j} \to \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{i}$. By the coset construction:

$$\mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{j} = \bigoplus H^{\ell}_{jj'} \otimes H^{\ell+2}_{j'} \text{ and } \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{i} = \bigoplus H^{\ell}_{ii'} \otimes H^{\ell+2}_{i'}$$

Let $p_{i'}$, $p_{j'}$ be the projection on $H^{\ell}_{ii'} \otimes H^{\ell+2}_{i'}$ and $H^{\ell}_{jj'} \otimes H^{\ell+2}_{j'}$.
Let $\eta \in H^{\ell}_{ii'}$, $\xi \in H^{\ell}_{jj'}$ be non-zero fixed L_0-eigenvectors.
Let $\phi(v, z) : H^{\ell+2}_{j'} \to H^{\ell+2}_{i'}$, defined by: $\forall \eta' \in H^{\ell+2}_{i'}$ and $\forall \xi' \in H^{\ell+2}_{j'}$,

$$(p_{i'}\Phi(v, z)p_{j'}.(\xi \otimes \xi'), \eta \otimes \eta') = (\phi(v, z).\xi', \eta').$$

Now, by compatibility condition for $LSU(2)$:

$$[X(n), \Phi(v, z)] = z^n\Phi(X.v, z) \text{ and } [L_n, \Phi(v, z)] = z^n[z\tfrac{d}{dz} + (n+1)h^{\ell}_{1/2}]\Phi(v, z)$$

Now, $X(n)$ and L_n commute with $p_{i'}$, $p_{j'}$ and z^r with $s \in \mathbb{Q}$, then, by easy manipulation we see that, up to multiply by a rational power of z:

$$[X(n), \phi(v,z)] = z^n(\phi(X.v,z)) \text{ and}$$
$$[L_n, \phi(v,z)] = z^n[z\tfrac{d}{dz} + (n+1)h_{1/2}^{\ell+2}](\phi(v,z))$$

By compatibility and uniqueness theorem, $\exists s \in \mathbb{Q}$ such that $z^s\phi(v,z)$ is the spin $1/2$ and level $\ell+2$ primary field $\phi_{i'j'}^{\ell+2}(v,z)$ (up to a multiplicative constant) of $LSU(2)$. The power s can be compute using lemma 3.44. It follows that $p_{i'}\Phi(v,z)p_{j'} = \phi_{i'j'}^{\ell+2}(v,z) \otimes \rho(z)$. Now, $h_{\frac{1}{2},\frac{1}{2}}^{\ell} = h_{\frac{1}{2}}^{\ell} - h_{\frac{1}{2}}^{\ell+2}$, it follows that up to multiply by a rational power of z:

$$[L_n, \rho(z)] = z^n[z\tfrac{d}{dz} + (n+1)h_{\frac{1}{2},\frac{1}{2}}^{\ell}]\rho(z)$$

We verify also, using the explicit formula for G_r that:

$$[G_{-1/2}, \rho(z)] = z^{-r-1/2}[G_r, \rho(z)]$$

Finally, by compatibility condition and uniqueness, $\exists s' \in \mathbb{Q}$ such that $z^{s'}\rho(z)$, is the charge $(1/2,1/2)$ primary field $\phi_{ii'jj'}^{\frac{1}{2}\frac{1}{2},\ell}(z)$ of $\mathfrak{Vir}_{1/2}$ between $H_{jj'}^{\ell}$ and $H_{ii'}^{\ell}$ (up to a multiplicative constant). Finally by lemma 3.44:

$$p_{i'}[I \otimes \phi_{ij}^{\frac{1}{2}\ell}(v,z)]p_{j'} = C.z^{-C_{ii'jj'}^{kk'}}\phi_{i'j'}^{\ell+2}(v,z) \otimes \phi_{ii'jj'}^{\frac{1}{2}\frac{1}{2},\ell}(z)$$

the value of σ follows using characterization: integer and half-integer moded.

Now, the constant C is possibly zero. So, we will prove it's non-zero for the annonced constructible fields:

If it exists j' such that, $\Phi(v,z)p_{j'} = 0$ $\forall v$, then, $\forall u \in L_{jj'}^{\ell} \otimes L_{j'}^{\ell+2}$, $\Phi(v,z)u = 0$, but, by commutation relation with $I \otimes \psi(x,r)$ and $X(n) \otimes I$, it follows by irreducibility that $u \neq 0$ is cyclic and $\Phi(v,z)u' = 0$ $\forall u' \in \mathcal{F}_{NS}^{\mathfrak{g}} \otimes L_j^{\ell}$. Then, $\Phi(v,z) = 0$ contradiction. So, $\forall j'$, $\Phi(v,z)p_{j'} \neq 0$, so it exists i' such that $p_{i'}\Phi(v,z)p_{j'} \neq 0$. By the beginning of the proof, a necessary condition for i' is that $\phi_{i'j'}^{\frac{1}{2}\ell+2}$ is a non-zero primary field of $LSU(2)$. We will prove that this condition is also sufficient. For now, we know that for this i', $\phi_{ii'jj'}^{\frac{1}{2}\frac{1}{2},\ell}$ is a non-zero primary field of $\mathfrak{Vir}_{1/2}$.

Now, $\forall i'$ let $\rho_{ii'jj'}^{\frac{1}{2}\frac{1}{2},\ell}$ a multiple (possibly zero) of $\phi_{ii'jj'}^{\frac{1}{2}\frac{1}{2},\ell}$, such that:

$$\Phi(v,z) = \Phi_{ij}(v,z) = \sum \rho_{ii'jj'}^{\frac{1}{2}\frac{1}{2},\ell}(z) \otimes \phi_{i'j'}^{\frac{1}{2},\ell+2}(v,z)$$

Now, $(\Phi_{ij}(u,z)\Phi_{jk}(v,w)\Omega_{jj'kk'} \otimes \Omega_{j'k'}, \Omega_{jj'ii'} \otimes \Omega_{j'i'})$
$= \sum(\rho_{ii'jj'}^{\frac{1}{2}\frac{1}{2},\ell}(z)\rho_{jj'kk'}^{\frac{1}{2}\frac{1}{2},\ell}(w)\Omega_{jj'kk'}, \Omega_{jj'ii'}).(\phi_{i'j'}^{\frac{1}{2},\ell+2}(u,z)\phi_{j'k'}^{\frac{1}{2},\ell+2}(v,w)\Omega_{j'k'}, \Omega_{j'i'})$
We can write it as a relation between reduced 4-point function:

$$F_j(\zeta) = \sum f_{j'}(\zeta) h_{jj'}(\zeta)$$

We return in the context of reminder 3.21 and 3.26: F_j and $f_{j'}$ are holomorphic function from $\mathbb{C}\backslash[0,\infty[$ to W. Let $v_{j'} \in W$ such that $g(\zeta)v_{j'} = \zeta^{\mu_{j'}} f_{j'}(\zeta)$. We apply the gauge transformation $g(\zeta)^{-1}$ on the previous equality:

$$g(\zeta)^{-1} F_j(\zeta) = \sum \zeta^{-\mu_{j'}} v_{j'} h_{jj'}(\zeta)$$

It follows that $h_{jj'}$ is holomorphic on $\mathbb{C}\backslash[0,\infty[$, we get a formula for it:

$$h_{jj'}(\zeta) = C.\zeta^{\mu_{j'}} (g(\zeta)^{-1} F_j(\zeta), v_{j'})$$

with C a non-zero constant.

This formula gives exactly the duality for braiding discovered by Tsuchiya-Nakanishi [22]. Then by reminder 3.26, the braiding matrix for the $\rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z)$ is the product of the braiding matrix for $LSU(2)$ at spin $1/2$ and level ℓ, times the transposed of the inverse of the braiding matrix for $LSU(2)$ at spin $1/2$ and level $\ell+2$. All the coefficients are non-zero. Now, suppose that $\rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z) = 0$, with $\phi^{\frac{1}{2},\ell}_{ij}$ and $\phi^{\frac{1}{2},\ell+2}_{i'j'}$ constructible then:

$$0 = \rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z) \rho^{\frac{1}{2}\frac{1}{2},\ell}_{jj'ii'}(w) = \sum \lambda_{kk'} \rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'kk'}(w) \rho^{\frac{1}{2}\frac{1}{2},\ell}_{kk'ii'}(z)$$

with all braiding coefficients non-zero. But as we see previously by irreducibility, the right side admits at least a non-zero term, contradiction.

For the braiding between charge $(0,1)$ and charge $(1/2,1/2)$ primary fields, we do the same starting with the Neveu-Schwarz fermion field $\psi(u,z) \otimes I$ commuting with $I \otimes \phi^{\frac{1}{2}\ell}_{ij}(v,w)$. We find also that every possible braiding coefficients are non-zero. The result follows. **End of the proof.**

Remark 3.53. *As a consequence of remark 3.24, we know that such a braiding exists for $(\gamma_1,\gamma_2) = (\beta,\beta)$, but we don't know if every coefficients $\mu_{rr'}$ are non-null.*

Proposition 3.54. *The primary fields of charge α or β are bounded and identifying the L^2-completion of $\mathcal{F}^\sigma_{\lambda,\mu}$ with $L^2(\mathbb{S}^1)e^{\sigma i\theta/2} \oplus L^2(\mathbb{S}^1)e^{(1-\sigma)i\theta/2}$, we obtain $\phi(f)$ for $f \in L^2(\mathbb{S}^1)e^{\sigma i\theta/2}$, $\theta(g)$ for $g \in L^2(\mathbb{S}^1)e^{(1-\sigma)i\theta/2}$ with:*

$$\|\phi(f)\| \leq K\|f\|_2 \quad \textit{and} \quad \|\theta(g)\| \leq K'\|g\|_2$$

Proof. The primary fields of charge α or β are constructibles, and the compressions of fermions are bounded operators. $\square$

Corollary 3.55. *(Localised braiding relation) Let $f \in L^2_I(\mathbb{S}^1)e^{\sigma i\theta/2}$ and $g \in L^2_J(\mathbb{S}^1)e^{\sigma i\theta/2}$, with I, J be two disjoint proper intervals of $\mathbb{S}^1$. Using an argument of convolution (as [25] p 516), we can write the following localised braiding relations:*

$$\phi^{\gamma_1 \ell}_{ii'jj'}(f)\phi^{\gamma_2 \ell}_{jj'kk'}(g) = \sum \mu_{rr'}\phi^{\gamma_2 \ell}_{ii'rr'}(e_\lambda g)\phi^{\gamma_1 \ell}_{rr'kk'}(\bar{e}_\lambda f) \text{ with } \mu_{rr'} \neq 0.$$

with $e_\lambda = e^{i\lambda\theta}$, $\lambda = h^\ell_{ii'} + h^\ell_{kk'} - h^\ell_{jj'} - h^\ell_{rr'}$ and (γ_1, γ_2) as previously.

3.4 Application to irreducibility

Definition 3.56. *Let $\mathcal{M}$, $\mathcal{N} \subset B(H)$ be von Neumann algebra, then, $\mathcal{M} \vee \mathcal{N}$ is the von Neumann algebra generated by $\mathcal{M}$ and $\mathcal{N}$.*

Notation 3.57. *We simply note $\phi^k_{ij}(f)$ for primary field of charge k for $\mathfrak{Vir}_{1/2}$; the charge c_m is fixed and $i = 0$ significate $i = (0, 0)$.*

Proposition 3.58. *The chains of constructible primary fields of the form:*

$$\phi^\alpha_{0i_1}(f_1)\phi^\alpha_{i_1i_2}(f_2)...\phi^\alpha_{i_{r-1}i_r}(f_r)\phi^\alpha_{i_r0}(f_{r+1}) \text{ with } \alpha = (\tfrac{1}{2}, \tfrac{1}{2}) \text{ and } f_i \text{ on } I.$$

are bounded operators and generate the von Neumann algebra $\mathcal{N}^\ell_{00}(I)$.

Proof. By corollary 2.84 and proposition 3.43. $\square$

Remark 3.59. *Let σ_t be the geometric modular action described on reminder 2.60. Let $\psi^k_{ij}(f)$ be a bounded primary field of charge k concentrated on a proper interval J. Let $\sigma_t(\psi^\alpha_{ij}(f)) := \pi_i(\varphi_t)\psi^\alpha_{ij}(f)\pi_j(\varphi_t)^\star = \psi^\alpha_{ij}(u_t.f)$ by the covariance relations. Then, $\sigma_t(\psi^\alpha_{ij}(f))$ is a primary field concentrated on $\varphi_t(J) \to \{1\}$ (when $t \to \infty$).*

Reminder 3.60. *(Cancellation theorem) If a unitary representation of a connected semisimple non-compact group with finite center has no fixed vectors, then its matrix coefficients vanish at ∞. We can find a proof on Zimmer's book [30]. For example, $G = SU(1,1) \simeq SL(2,\mathbb{R})$ (non-compact) is implemented on the irreducible positive energy representations H of $\mathfrak{Vir}_{1/2}$, which give a unitary representation of a central cyclic extension $\mathcal{G}$ of G, whose*

Lie algebra is generated by L_{-1}, L_0 and L_1. But if $\xi \in H$, $L_0\xi = 0$ implies immediatly that $H = H_0$ and $\xi = \Omega$ (up to a multiplicative constant). So G admits no fixed vectors outside of the vacuum. But the modular operators U_t go to ∞ when $t \to \infty$. Then, their matrix coefficients vanish at ∞. In our case, we can prove the cancellation theorem directly, because H decomposes into a direct sum of irreducible positive energy representation of $\mathcal{G}$ and each summands is a discrete series representation of $\mathcal{G}$, so can be realized as a subrepresentation of $L^2(\mathcal{G})$, and then has matrix coefficient tending to zero at ∞ (see Pukanszky [17]).

Proposition 3.61. *(Generically non-zero) Let $a = \phi^{\alpha}_{\alpha 0}(f)$ and $b = \phi^{\alpha}_{0\alpha}(g)$ with f, g on proper intervals. Then, $(ba\Omega, \Omega)$ is non-zero in general.*

Proof. Let $a = \phi^{\alpha}_{\alpha 0}(f) \neq 0$ and R_θ be the quantized rotation action: $R_\theta = e^{iL_0\theta}$ (see remark 2.61). Let $b_\theta = R^\star_\theta a^\star R_\theta$. We suppose that $(b_\theta a\Omega, \Omega) = 0$ for $|\theta - \theta_1| \leq \varepsilon$ with θ_1 fixed and $\varepsilon > 0$. Then $(R^\star_\theta a^\star R_\theta a\Omega, \Omega) = 0$. But $L_0\Omega = 0$ on the vacuum representation. Then, $R_\theta\Omega = \Omega$ and $(R_\theta a\Omega, a\Omega) = 0$. Now, by positive energy of the representation $a\Omega = \sum_{n\in\frac{1}{2}\mathbb{N}} \xi_n$ (coming from the orthogonal decomposition for L_0) and $\|a\Omega\|^2 = \sum \|\xi_n\|^2$. Now, with $z = e^{i\theta/2}$, $(R_\theta a\Omega, a\Omega) = \sum_{n\in\mathbb{N}} z^n \|\xi_{n/2}\|^2 = f(z)$, let $g(z) = f(e^{-i\theta_1/2}z)$. Then, g extends to a continuous function on the closed unit disc, holomorphic in the interior and vanishing on the unit circle near $\{1\}$. By the Schwarz reflection principle and the Cayley transfrom, g must vanishes identically in z. So, $(R_0 a\Omega, a\Omega) = \|a\Omega\|^2 = 0$. Then $a\Omega = 0$, so $a^\star a\Omega = 0$. But Ω is a separating vector on the von Neumann algebra, so $a^\star a = 0$, and $a = 0$, contradiction. ☐

Proposition 3.62. *(Leading term in OPE of primary fields)*
Let I be a proper interval of $\mathbb{S}^1$, and I_1, I_2 be subintervals be subintervals obtained by removing a point. Let $a_{\nu\mu}$ and $b_{\mu\nu}$ be non-zero primary fields of charge α, localised in I_1 and I_2 respectively, then $\sigma_t(a_{\nu\mu}b_{\mu\nu}) \to^w Id_{H_i}$ (up to a multiplicative constant).

Proof. We adapt to $\mathfrak{Vir}_{1/2}$, a proof of A. Wassermann [28] for $LSU(2)$.

Without a loose of generality, we can take $\{1\} \in \bar{I}_1 \cap \bar{I}_2$. Let a and b be generic primary fields of charge α concentrate on I_2 and I_1 respectively.

(1)We first prove that $\sigma_t(a_{0\alpha}b_{\alpha 0}) \to^w C$ non-zero constant:
$\|\sigma_t(a_{0\alpha}b_{\alpha 0})\|$ is clearly bounded, then by the weak compacity of the unit

ball, it exists a sequence t_n such that $\sigma_{t_n}(a_{0\alpha}b_{\alpha 0}) \to^w T$. By the remark 3.59, $\sigma_{t_n}(b_{0\alpha}a_{\alpha 0})$ is concentrated on J_n with $\bigcap J_n = \{1\}$. We obtain that T supercommutes with $\bigvee \mathcal{N}_{00}^{\ell}(J_n^c)$. By Araki-Haag duality, $(\bigvee \mathcal{N}_{00}^{\ell}(J_n^c))^{\natural} = \bigcap \mathcal{N}_{00}^{\ell}(J_n)) = \mathcal{N}_{00}^{\ell}(\{1\}) = \mathbb{C}$. Then $T \in \mathbb{C}Id$. Now, $(\sigma_{t_n}(a_{0\alpha}b_{\alpha 0})\Omega, \Omega) = (a_{0\alpha}b_{\alpha 0}\Omega, \Omega)$ because $\pi_0(U_t)\Omega = \Omega$ (see remark 2.61). Now $(a_{0\alpha}b_{\alpha 0}\Omega, \Omega) = k$ generically non-zero (proposition 3.61) and $T = kId$. Now, k is independant on the sequence (t_n), so $\sigma_t(a_{0\alpha}b_{\alpha 0}) \to^w k.Id \neq 0$.

(2) We now prove that $\sigma_t(a_{\gamma\alpha}b_{\alpha 0}) \to^w 0$ if $\gamma \neq 0$.
Idem, it exists a sequence t_n such that $X_n = \sigma_{t_n}(a_{\gamma\alpha}b_{\alpha 0}) \to^w T$. Let ξ be a finite energy vector in H_γ, then $(X_n\Omega, \xi) = (\pi_\gamma(U_{t_n})a_{\gamma\alpha}b_{\alpha 0}\Omega, \xi) = ((\pi_\gamma(U_{t_n})\eta, \xi) \to 0$ when $t_n \to \infty$ by the cancellation theorem (reminder 3.60)Then, $T\Omega = 0$, so $T^\star T\Omega = 0$. But Ω is a separating vector on the von Neumann algebra, so $T^\star T = 0$ and $T = 0$. Now, the 0 is independent of the choice of the sequence, then: $\sigma_t(a_{\gamma\alpha}b_{\alpha 0}) \to^w 0$.

(3) We prove that if $a_{\nu\mu} \neq 0$, then $\sigma_t(a_{\nu\mu}b_{\mu\nu}) \to^w C'$ non-zero constant:
Idem, it exists a sequence such that $\sigma_{t_n}(a_{\nu\mu}b_{\mu\nu}) \to^w R$. Now, let $y_{\nu 0} = x_{\nu\lambda_1}x_{\lambda_1\lambda_2}...x_{\lambda_r 0}$ be a chain between ν and 0 with the minimal number of primary fields of charge α, concentrate on a proper closed K interval out of $\{1\}$. Then for t sufficiently large, we can apply the braiding formulas on $\sigma_t(a_{\nu\mu}b_{\mu\nu})y_{\nu 0}$. We obtain necessarily $\sigma_t(a_{\nu\mu}b_{\mu\nu})y_{\nu 0} = \sum_{\gamma\neq 0} A_\gamma\sigma_t(a_{\gamma\alpha}b_{\alpha 0}) + \lambda y_{\nu 0}\sigma_t(a_{0\alpha}b_{\alpha 0})$, with $\lambda \neq 0$, A_γ a linear sum of non-minimal chains between ν and γ (note that in general, there are many ways to go between 0 and ν minimally, but by the structure of the braiding rules, only the way chosen for $y_{\nu 0}$ can appear at the end). Now, by (1) and (2), the previous equality (with $t = t_n$) weakly converge to $Ry_{\nu 0} = \lambda y_{\nu 0}C = \lambda C y_{\nu 0}$ with λC a non-zero constant. Now, $R \in \mathcal{N}_\nu^\ell(K^c)$, then $Ry_{\nu 0} = y_{\nu 0}\pi_0(R) = \lambda C y_{\nu 0}$. Now $\sigma_t(y_{\nu 0})$ is also a minimal chain of charge α between ν and 0, concentrate on a proper closed interval out of $\{1\}$, so $\sigma_t(y_{\nu 0})\pi_0(R) = C'\sigma_t(y_{\nu 0})$ with C' a non-zero constant. Then $\sigma_t(y_{\nu 0})^\star\sigma_t(y_{\nu 0})\pi_0(R) = C'\sigma_t(y_{\nu 0})^\star\sigma_t(y_{\nu 0})$. But $\sigma_t(y_{\nu 0})^\star\sigma_t(y_{\nu 0}) = \sigma_t(y_{\nu 0}^\star y_{\nu 0}) \to_w k.Id \neq 0$ as for (1). So $\pi_0(R) = C' = R$. $\square$

Proposition 3.63. *(von Neumann density) Let I be a proper interval of $\mathbb{S}^1$, and I_1, I_2 be subintervals such that $I = I_1 \cup I_2$.*

$$\mathcal{N}_{ij}^\ell(I_1) \vee \mathcal{N}_{ij}^\ell(I_2) = \mathcal{N}_{ij}^\ell(I).$$

Proof. By the local equivalence for $\mathfrak{Vir}_{1/2}$ (see section 2.7), we only need to prove the result on the vacuum. By proposition 3.58 we only need to work with chains. Consider the chain $\phi^{\alpha}_{0i_1}(f_1)\phi^{\alpha}_{i_1i_2}(f_2)...\phi^{\alpha}_{i_{r-1}i_r}(f_r)\phi^{\alpha}_{i_r0}(f_{r+1}) \in \mathcal{N}^{\ell}_{00}(I)$, with $f_k \in L^2_I(\mathbb{S}^1)$. Now, $f_k = f_k^{(1)} + f_k^{(2)}$, with $f_k^{(i)}$ concentrated on I_i. Now, a primary field $\phi^k_{ij}(f)$ is linear in f, so, we can develop the chain into a sum of chains of primary filed localized exclusively on I_1 or I_2. Next, applying the braiding relations, we can obtain a linear combination of chains, on which the primary field localized on I_1 and I_2 are separated; generically of the form:

$$\phi^{\alpha}_{0j_1}(g_1)\phi^{\alpha}_{j_1j_2}(g_2)...\phi^{\alpha}_{j_{s-1}j_s}(g_{s-1})\phi^{\alpha}_{j_sj_{s+1}}(h_{s+1})...\phi^{\alpha}_{j_{r-1}j_r}(h_r)\phi^{\alpha}_{j_r0}(h_{r+1})$$

with g_k and h_k concentrate on I_1 and I_2 respectively. Now, if $j_s = 0$, then, the previous chain is a product $a.b$ with $a \in \mathcal{N}^m_{11}(I_1)$ and $b \in \mathcal{N}^m_{11}(I_2)$.
Else, if $j_s \neq 0$, using the previous proposition step by step, we see that the chain is the weak limit of chains with 0 on the middle, the result follows. $\square$

Lemma 3.64. *(Covering lemma) Let (I_n) be a covering of $\mathbb{S}^1$ by open proper intervals. Then $\mathfrak{Vir}_{1/2}(\mathbb{S}^1)$ is the linear span of the $\mathfrak{Vir}_{1/2}(I_n)$. And so $\bigvee \pi(\mathfrak{Vir}_{1/2}(I_n))'' = \pi(\mathfrak{Vir}_{1/2}(\mathbb{S}^1))'' = B(H)$.*

Proof. With a partition of the unity. $\square$

Theorem 3.65. *Let I be a proper interval of $\mathbb{S}^1$, then, the Jones-Wassermann subfactor $\mathcal{N}^{\ell}_{ij}(I) \subset \mathcal{N}^{\ell}_{ij}(I)^{\natural}$ is irreducible, i.e. $\mathcal{N}^{\ell}_{ij}(I)^{\natural} \cap \mathcal{N}^{\ell}_{ij}(I^c)^{\natural} = \mathbb{C}$.*

Proof. Let I_1, I_2 be two proper subintervals of I obtained by removing a point. Let $J_1 = I$, $J_2 = \overline{I_1 \cup I^c}$ and $J_3 = \overline{I^c \cup I_2}$. Let $\mathcal{M} = \mathcal{N}^{\ell}_{ij}(I) \vee \mathcal{N}^{\ell}_{ij}(I^c)$, then $\mathcal{N}^{\ell}_{ij}(I)$,$\mathcal{N}^{\ell}_{ij}(I^c)$,$\mathcal{N}^{\ell}_{ij}(I_1)$ and $\mathcal{N}^{\ell}_{ij}(I_2) \subset \mathcal{M}$. By von Neumann density, $\mathcal{N}^{\ell}_{ij}(J_2) = \mathcal{N}^{\ell}_{ij}(I_1) \vee \mathcal{N}^{\ell}_{ij}(I^c) \subset \mathcal{M}$, and idem $\mathcal{N}^{\ell}_{ij}(J_3) \subset \mathcal{M}$. Let K_1, K_2, K_3 be open subintervals of J_1, J_2 and J_3 such that $K_1 \cup K_2 \cup K_3 = \mathbb{S}^1$. Now, $N^{\ell}_{ij}(K_1) \vee N^{\ell}_{ij}(K_2) \vee N^{\ell}_{ij}(K_3) \subset \mathcal{M}$, but $N^{\ell}_{ij}(K_1) \vee N^{\ell}_{ij}(K_2) \vee N^{\ell}_{ij}(K_3) = B(H^{\ell}_{ij})$ by covering lemma. So $\mathcal{M} = B(H^{\ell}_{ij})$ and $\mathbb{C} = \mathcal{M}^{\natural} = \mathcal{N}^{\ell}_{ij}(I)^{\natural} \cap \mathcal{N}^{\ell}_{ij}(I^c)^{\natural}$. $\square$

4 Connes fusion and subfactors

4.1 Recall on subfactors

See the book [8] for a complete introduction to subfactors.

Definition 4.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebra, then, an inclusion $\mathcal{N} \subset \mathcal{M}$ is called a subfactor.*

Reminder 4.2. *A factor $\mathcal{M}$ of type II admits a canonical trace tr. The image of tr on the subset of projection of $\mathcal{M}$ is $[0, 1]$ or $[0, \infty]$.
Then, $\mathcal{M}$ is said to be a factor of type II_1 or II_∞.*

Reminder 4.3. *(Basic construction) Let the subfactor $\mathcal{N} \subset \mathcal{M}$, with $\mathcal{M}$ and $\mathcal{N}$ II_1 factors. Let tr be the trace on $\mathcal{M}$, then, it admit the following inner product: $(x, y) := tr(xy^\star)$. Let $H = L^2(\mathcal{M}, tr)$ and $L^2(\mathcal{N}, tr)$ be the L^2-completions of $\mathcal{M}$ and $\mathcal{N}$. Let $e_\mathcal{N}$ be the orthogonal projection of $L^2(\mathcal{M}, tr)$ onto $L^2(\mathcal{N}, tr)$.
Let $\langle \mathcal{M}, e_\mathcal{N} \rangle = (\mathcal{M} \cup \{e_\mathcal{N}\})'' \subset B(H)$. It admit a trace called $tr_{\langle \mathcal{M}, e_\mathcal{N} \rangle}$.
The tower $\mathcal{N} \subset \mathcal{M} \subset \langle \mathcal{M}, e_\mathcal{N} \rangle$ is called the basic construction.*

Reminder 4.4. *(Index of subfactors) Let the previous subfactor $\mathcal{N} \subset \mathcal{M}$.
Then we can define its index $[\mathcal{M} : \mathcal{N}] = (tr_{\langle \mathcal{M}, e_\mathcal{N} \rangle}(e_\mathcal{N}))^{-1} \in [1, \infty]$.
The index admits another definition as the von Neumann dimension (see [8]) of the $\mathcal{N}$-module $H = L^2(\mathcal{M}, tr)$, ie $[\mathcal{M} : \mathcal{N}] = dim_\mathcal{N}(H)$.*

Reminder 4.5. *(Jones' theorem, see [6]) Every possible index of II_1-subfactors:*

$$\{4cos^2(\tfrac{\pi}{m}) | m = 3, 4, ...\} \cup [4, \infty]$$

In the continuation of the basic construction, we can build a graph from a subfactor, called its principal graph. If the subfactor admits a finite index then the square of the norm of the matrix of its principal graph is exactly the index. Now, this matrix admits only integers values, and a theorem of Kronecker said that the norm of an integer valued matrix is in $\{2cos(\frac{\pi}{m}) | m = 3, 4, ...\} \cup [2, \infty]$. Finally, it's proved that every possible such norms are realized from subfactors.

Definition 4.6. *A subfactor of finite index $\mathcal{M} \subset \mathcal{N}$ is said to be irreducible if either of the following equivalent conditions are satisfied:*

(a) $L^2(\mathcal{M})$ is irreducible as an $\mathcal{N}$-$\mathcal{M}$-bimodule.

(b) The relative commutant $\mathcal{N}' \cap \mathcal{M}$ is $\mathbb{C}$.

4.2 Bimodules and Connes fusion

Definition 4.7. *If $\mathcal{M}$, $\mathcal{N}$ are $\mathbb{Z}_2$-graded von Neumann algebra, a $\mathbb{Z}_2$-graded Hilbert space H is said to be a $\mathcal{M}$-$\mathcal{N}$-bimodule if:*

(a) H is a left $\mathcal{M}$-module.

(b) H is a right $\mathcal{N}$-module.

(c) the action of $\mathcal{M}$ and $\mathcal{N}$ supercommute; i.e.,
$\forall m \in \mathcal{M}$, $n \in \mathcal{N}$, $\xi \in H$, $(m.\xi).n = (-1)^{\partial m \partial n} m.(\xi.n)$.

Definition 4.8. *Let $\Omega \in H_0$ be a vacuum vector, then H_0 is a $\mathcal{M}$-$\mathcal{M}$ bimodule, because by Tomita-Takesaki theory, $J\mathcal{M}J = \mathcal{M}'$, by lemma 2.24, $\mathcal{M}^\natural = \kappa\mathcal{M}'\kappa^\star \simeq \mathcal{M}' \simeq \mathcal{M}^{opp}$. Now, $y^\star x^\star = (xy)^\star$ and $\mathcal{M}^{opp}$ is the opposite algebra: $a \times b = b.a$. Then $x.(\xi.y) := x(\kappa J y^\star J \kappa^\star)\xi$ gives the bimodule action.*

Definition 4.9. *(Intertwinning operators) Let X, Y be $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodules, $\mathcal{X} = Hom_{-\mathcal{M}}(H_0, X)$ and $\mathcal{Y} = Hom_{\mathcal{M}-}(H_0, Y)$ be the space of bounded operators that superintertwin the left (resp. the right) action of $\mathcal{M}$.*

Lemma 4.10. *Consider the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$, we define a pre-inner product by:*

$$(x_1 \otimes y_1, x_2 \otimes y_2) = (-1)^{(\partial x_1 + \partial x_2)\partial y_2}(x_2^\star x_1 y_2^\star y_1 \Omega, \Omega)$$

Proof. As for [25] p 525-526. □

Definition 4.11. *The L^2-completion is called the Connes fusion between X and Y, and noted $X \boxtimes Y$, naturally a $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodule.*

Lemma 4.12. *There are canonical unitary isomorphism*

$$H_0 \boxtimes X \simeq X \simeq X \boxtimes H_0.$$

Proof. If $Y = H_0$, the unitary $X \boxtimes H_0 \to X$ is given by $x \otimes y \mapsto xy\Omega$, and the unitary $H_0 \boxtimes X \to X$ is given by $y \otimes x \mapsto (-1)^{\partial x \partial y} xy\Omega$. □

Lemma 4.13. *$\mathcal{X}$ can be seen as a dense subspace of X via $x \leftrightarrow x\Omega$.*

Proof. $\mathcal{X} = \mathcal{X}.\pi_0(\mathcal{M}(I^c))$, so by Reeh-Schlieder $\mathcal{X}\Omega$ is dense in $\mathcal{X}H_0$.
Now, $\mathcal{X}H_0 = [\pi_X(\mathcal{M}(I^c))\mathcal{X}].[\pi_0(\mathcal{M}(I)).H_0] = \pi_X(\mathcal{M}(I^c).\mathcal{M}(I))\mathcal{X}\mathcal{H}_0$
$= \pi_X(\langle \mathcal{M}(I^c).\mathcal{M}(I)\rangle_{lin})\mathcal{X}\mathcal{H}_0$. But, because $\mathcal{M}(I^c)$ and $\mathcal{M}(I)$ supercommute, the $\star$-algebra generated by $\mathcal{M}(I^c).\mathcal{M}(I)$ is exactly its linear span, then, $\pi_X(\langle \mathcal{M}(I^c).\mathcal{M}(I)\rangle_{lin})$ is weakly dense in $\pi_X(\mathcal{M}(I^c).\mathcal{M}(I))''$. So, by von Neumann density $\mathcal{X}H_0$ is dense in $\bigoplus B(H_i)\mathcal{X}H_0 = X$, with $X = \bigoplus H_i$. □

Lemma 4.14. *(Hilbert space continuity lemma)*
The natural map $\mathcal{X} \otimes \mathcal{Y} \to X \boxtimes Y$ extends canonically to continuous maps $X \otimes \mathcal{Y} \to X \boxtimes Y$ and $\mathcal{X} \otimes Y \to X \boxtimes Y$. In fact $\|x_i \otimes y_i\|^2 \leq \|x_i x_i^\star\| \sum \|y_i\Omega\|^2$ and $\|x_i \otimes y_i\|^2 \leq \|y_i y_i^\star\| \sum \|x_i\Omega\|^2$

Proof. As for [25] p 526. □

Lemma 4.15. $\boxtimes$ *is associative.*

Proof. As for [25] p 527. □

4.3 Connes fusion with H_α on $\mathfrak{Vir}_{1/2}$

Remark 4.16. *Note that the primary fields ϕ we consider are always the ordinary part and so even operators. In fact, we only need to consider even intertwiner operators because each odd intertwiner operator is the product of an even one and an odd operator on the vacuum local von Neumann algebra.*

Definition 4.17. *Let $\langle i, j\rangle := \{k \mid \phi_{ij}^k \neq 0 \}$.*

Recall that the primary field of charge $\alpha = (1/2, 1/2)$ are bounded. Let the graph $\mathcal{G}_\alpha$ with vertices $\{i\}$ and an edge between i and j if $j \in \langle \alpha, i\rangle$; then, α is a weak generator in the sense that the graph $\mathcal{G}_\alpha$ is connected. Let I be a non-trivial interval of $\mathbb{S}^1$, and let f and g be L^2-functions localized in I and I^c respectively. Recall that every possible braiding at charge α admits non-null coefficients, ie; $\phi_{ij}^\alpha(z)\phi_{jk}^\alpha(w) = \sum \lambda_l \phi_{il}^\alpha(w)\phi_{lk}^\alpha(z)$ with $\lambda_l \neq 0$ iff $l \in \langle \alpha, i\rangle \cap \langle \alpha, k\rangle$. Then, by the standard convolution argument: $\phi_{ij}^\alpha(f)\phi_{jk}^\alpha(g) = \sum \lambda_l \phi_{il}^\alpha(e_l g)\phi_{lk}^\alpha(\bar{e}_l f)$ with e_l the phase correction. We note $a_{0\alpha} = \phi_{0\alpha}^\alpha(f)$, $b_{\alpha 0} = \phi_{\alpha 0}^\alpha(g)$ called the principal part. We define the non-principal parts a_{ij} and b_{ij} such that they incorporate the phase correction in the braiding relations. Next, if $a_{ij} = \phi_{ij}^\alpha(h)$ then $a_{ij}^\star = \phi_{ji}^\alpha(\bar{h})$, so we note $\bar{a}_{ji} = a_{ij}^\star$:

Corollary 4.18. *(Braiding relations)*

$$b_{ij}a_{jk} = \sum \nu_l a_{il} b_{lk} \quad \textit{with } \nu_l \neq 0 \textit{ iff } l \in \langle \alpha, i \rangle \cap \langle \alpha, k \rangle$$

Corollary 4.19. *(Abelian braiding) If $\#(\langle \alpha, i \rangle \cap \langle \alpha, k \rangle) = 1$ then:*

$$b_{ij}a_{jk} = \nu a_{ij} b_{jk} \quad \textit{with } \nu \neq 0$$

Lemma 4.20. *The set of vectors of the form $\eta = (\eta_i)$ with, $\eta_i = \pi_i(x) b_{ij} \xi$, $i \in \langle \alpha, j \rangle$, $x \in \mathcal{M}(I^c)$ and $\xi \in H_j$, spans a dense subspace of $\bigoplus H_i$.*

Proof. By Reeh-Schlieder, choosing a non-null vector $v_j \in F_j$, $\pi_j(\mathcal{M}(I^c))v_j$ is dense in H_j. Now, by intertwining, $b_{ij}\pi_j(\mathcal{M}(I)) = \pi_i(\mathcal{M}(I))b_{ij}$. Then, if $b_{ij}v_j = 0$, then, b_{ij} vanishes on a dense subspace, and so by continuity, $b_{ij} = 0$, contradiction. So, $b_{ij}v_j \neq 0$. Now, clearly, the set of vector $\rho = (\rho_i)$, with $\rho_i = \pi_i(x)b_{ij}\pi_j(y)v_j$, $x \in \mathcal{M}(I^c)$ and $y \in \mathcal{M}(I)$, is a subset of the set of the lemma. Now, by intertwining $\rho_i = \pi_i(x)\pi_i(y)b_{ij}v_j$. Let $\pi = \bigoplus \pi_i$ and $w = (w_i)$, with $w_i = b_{ij}v_j \neq 0$. Then, the set of ρ is exactly $\pi(\mathcal{M}(I^c).\mathcal{M}(I)).w$. Next, because $\mathcal{M}(I^c)$ and $\mathcal{M}(I)$ commute, the linear span of $\pi(\mathcal{M}(I^c).\mathcal{M}(I))$ is weakly dense in $\pi(\mathcal{M}(I^c).\mathcal{M}(I))'' = \bigoplus B(H_i)$ by von Neumann density. So, the set spans a dense subspace of $(\bigoplus B(H_i))w = \bigoplus H_i$ because $w_i \neq 0$. □

Remark 4.21. $\bar{a}_{ij}.a_{ji} \in Hom_{\mathcal{M}(I^c)}(H_i, H_i) = \pi_i(\mathcal{M}(I^c))'$.
In particular, $\bar{a}_{0\alpha}.a_{\alpha 0} \in \pi_0(\mathcal{M}(I))$ by Haag-Araki duality.

Definition 4.22. *Let $|i|$ be the less number of edges from i to 0 in the connected graph $\mathcal{G}_\alpha$.*

Theorem 4.23. *(Transport formula)*

$$\pi_i(\bar{a}_{0\alpha}.a_{\alpha 0}) = \sum_{j \in \langle \alpha, i \rangle} \lambda_j \bar{a}_{ij}.a_{ji} \quad \textit{with } \lambda_j > 0.$$

Proof. We prove by induction on $|i|$. We suppose that:

$$\pi_i(\bar{a}_{0\alpha}.a_{\alpha 0}) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda_j \bar{a}_{ij}.a_{ji} \quad \text{and} \quad \pi_i(\bar{b}_{0\alpha}.b_{\alpha 0}) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda'_j \bar{b}_{ij}.b_{ji}$$

(1) Polarizing the second identity, we get:

$$\pi_i(\bar{b}_{0\alpha}.b'_{\alpha 0}) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda'_j \bar{b}_{ij}.b'_{ji}$$

Now, with $x \in \mathcal{M}(I^c)$ and $b'_{ij} = \pi_i(x)b_{ij}\pi_j(x)^\star$, we get:

$$\pi_i(\bar{b}_{0\alpha}.\pi_\alpha(x)b_{\alpha 0}\pi_0(x)^\star) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda'_j \bar{b}_{ij}.\pi_j(x).b_{ji}\pi_i(x)^\star$$

Now, $\pi_i(\pi_0(x)^\star) = \pi_i(x)^\star$, so you can simplify by $\pi_i(x)^\star$:

$$\pi_i(\bar{b}_{0\alpha}.\pi_\alpha(x)b_{\alpha 0}) = \sum_{j\in\langle\alpha,i\rangle} \lambda'_j \bar{b}_{ij}.\pi_j(x).b_{ji}$$

(2) Next, by (1) and the braiding relations, $\bar{a}_{ik}\pi_k(\bar{b}_{0\alpha}\pi_\alpha(x)b_{\alpha 0})a_{ki} = \pi_i(\bar{b}_{0\alpha}\pi_\alpha(x)b_{\alpha 0})\bar{a}_{ik}a_{ki} = \sum_j \sum_{l,s} \lambda'_j \nu_l \mu_s \bar{b}_{ij}\bar{a}_{jl}a_{ls}\pi_s(x)b_{si}$.
Let $y = \bar{a}_{ik}\pi_k(\bar{b}_{0\alpha}\pi_\alpha(x^\star x)b_{\alpha 0})a_{ki} = a^\star_{ki}\pi_k(b^\star_{\alpha 0}\pi_\alpha(x^\star x)b_{\alpha 0})a_{ki}$ clearly a positive operator, then, $\forall \xi \in H_i$, $(y\xi,\xi) \geq 0$. Then, with $\eta_s = \pi_s(x)b_{si}\xi$, we obtain:

$$\sum \lambda'_j \nu_l \mu_s (a_{ls}\eta_s, a_{lj}\eta_j) \geq 0$$

(3) We now show that this inequality is linear in η:
Let $\tilde{\eta} = \sum \eta^r$ with $\eta^r = (\eta^r_s)$, $\eta^r_s = \pi_s(x_r)b_{si}\xi_r$, $x_r \in \mathcal{M}(I^c)$ and $\xi_r \in H_i$. Idem, $Y = (y_{rt})$ with $y_{rt} = a^\star_{ik}\pi_k(b^\star_{\alpha 0}\pi_\alpha(x^\star_r x_t)b_{\alpha 0})a_{ik}$, is a positive operator-valued matrix, so that $\sum_{r,t}(y_{rt}\xi_t,\xi_r) \geq 0$, which is exactly the inequality $\sum \lambda'_j \nu_l \mu_s (a_{ls}\tilde{\eta}_s, a_{lj}\tilde{\eta}_j) \geq 0$, and the linearity follows.
(4) Next, by lemma 4.20, the set of such η span a dense subspace of $\bigoplus H_s$, then, by linearity and continuity, the inequality runs $\forall \eta \in \bigoplus H_s$.
In particular, taking all but one η_j equal to zero, we obtain $\forall \eta_j \in H_j$:

$$\lambda'_j \mu_j \sum_l \nu_l \|a_{lj}\eta_j\|^2 \geq 0$$

(5) Now, restarting from $\tilde{Y} = (\pi_k(z_u)^\star Y \pi_k(z_v))$ with $z_u \in \mathcal{M}(I)$, we obtain:

$$\lambda'_j \mu_j \sum_l \nu_l \|\rho_l\|^2 \geq 0 \quad \forall (\rho_l) \in \bigoplus H_l$$

Choosing all but one ρ_l equal to zero, we have $\lambda'_j \nu_l \mu_j > 0$, and so $\nu_l \mu_j > 0$.
(6) Let $Z = (z_{rt})$, with $z_{rt} = b^\star_{ji}\pi_j(a^\star_{\alpha 0}a_{\alpha 0})\pi_j(x^\star_r x_t)b_{ji}$, and $x_r \in \mathcal{M}(I^c)$.
Z is a positive operator-valued matrix, so by the same process, induction and intertwining, we get:

$$\sum \lambda_k \nu_l \mu_s (a_{ls}\eta_s, a_{lj}\eta_j) = (\pi_j(a^\star_{\alpha 0}a_{\alpha 0})\eta_j, \eta_j)$$

Since it's true for all $\eta_s \in \bigoplus H_s$, all the term with $s \neq j$ are null:

$$(\pi_j(a^\star_{\alpha 0}a_{\alpha 0})\eta_j, \eta_j) = \sum \lambda_k \nu_l \mu_j (a_{lj}\eta_j, a_{lj}\eta_j)$$

But, we know that $\nu_l \mu_j > 0$, then, by induction hypothesis;

$$\pi_j(\bar{a}_{0\alpha}a_{\alpha 0}) = \sum \Lambda_l \bar{a}_{jl}a_{lj}, \text{ with } \Lambda_l > 0$$

The result follows because α is a weak generator and $j \in \langle\alpha, i\rangle$. $\square$

Corollary 4.24. *(Connes fusion for charge α)*

$$H_\alpha \boxtimes H_i = \bigoplus_{j\in\langle\alpha,i\rangle} H_j$$

Proof. Let $\mathcal{X}_0 \subset Hom_{\mathcal{M}(I^c)}(H_0, H_\alpha)$, be the linear span of intertwiners $x = \pi_\alpha(h)a_{\alpha 0}$, with $h \in \mathcal{M}(I)$ and $a_{\alpha 0}$ a primary field localised in I. Since $x\Omega = (\pi_\alpha(h)a_{\alpha 0}\pi_0(h)^\star)\pi_0(h)\Omega$ with h unitary, and $\pi_\alpha(h)a_{\alpha 0}\pi_0(h)^\star$ also a primary field, it follows by the Reeh-Schlieder theorem (and by the fact that the unitary operators generate the von Neumann algebra) that $\mathcal{X}_0\Omega$ is dense in $\mathcal{X}_0 H_0$. Now, using the von Neumann density in the same way that for the lemma 4.13, $\mathcal{X}_0\Omega$ is also dense in H_α. Let $x = \sum \pi_\alpha(h^{(r)})a_{\alpha 0} \in \mathcal{X}_0$, $x_{ji} = \sum \pi_j(h^{(r)})a_{ji}^{(r)}$ and $y \in \mathcal{Y} := Hom_{\mathcal{M}(I)}(H_0, H_i)$. By the transport formula: $(x^\star x y^\star y\Omega, \Omega) = (y^\star \pi_i(x^\star x)y\Omega, \Omega) = \sum \lambda_j \|x_{ji}y\Omega\|^2$. Now, polarising this identity, we get an isometry U of the closure of $\mathcal{X}_0 \otimes \mathcal{Y}$ in $H_\alpha \boxtimes H_i$ into $\bigoplus H_j$, sending $x\otimes y$ to $\bigoplus \lambda_j^{1/2} x_{ji}y\Omega$. By the Hilbert space continuity lemma, $\mathcal{X}_0 \otimes \mathcal{Y}$ is dense in $H_\alpha \boxtimes H_i$. Now, each a_{ji} can be non-zero, so by the unicity of the decomposition into irreducible, U is surjective and then a unitary operator. $\square$

Corollary 4.25. *(Commutativity for charge α)*

$$H_\alpha \boxtimes H_i = H_i \boxtimes H_\alpha$$

Proof. We prove in the same way that $H_i \boxtimes H_\alpha = \bigoplus_{j\in\langle\alpha,i\rangle} H_j$. $\square$

4.4 Connes fusion with H_β

Recall that $\beta = (0,1)$ and $\phi^\alpha_{\alpha,\beta}$ is non-zero.

$$\phi^\alpha_{ij}(z)\phi^\beta_{jk}(w) = \sum \lambda_l \phi^\beta_{il}(w)\phi^\alpha_{lk}(z) \text{ with } \lambda_l \neq 0 \text{ iff } l \in \langle\beta,i\rangle \cap \langle\alpha,k\rangle$$

Remark 4.26. *We proceed as previously: this braiding pass to the local primary field, we make principal and non-principal part incorporating the phase correction. Now, β is not a weak generator, so, to prove a transport formula, we prove by induction on $|i|$ that $a_{i0}c^\star_{\beta 0}c_{\beta 0} = [\sum \lambda_l c^\star_{li}c_{li}]a_{i0}$, with a_{i0} a chain of even primary field of charge α localised on I, (c_{ij}) even primary fields of charge β localised on I^c, and $\lambda_l \geq 0$ iff $l \in \langle\beta,i\rangle$. The proof uses the same arguments with positive operators... then by intertwining we obtain the following partial transport formula, and next, a partial fusion rules:*

Corollary 4.27. *(Transport formula)*

$$\pi_i(\bar{c}_{0\beta}.a_{\alpha 0}) = \sum_{j\in\langle\beta,i\rangle} \lambda_j \bar{c}_{ij}.c_{ji} \quad \textit{with } \lambda_j \geq 0.$$

Corollary 4.28. *(partial Connes fusion for β)*

$$H_\beta \boxtimes H_i \leq \bigoplus_{j\in\langle\beta,i\rangle} H_j$$

4.5 The fusion ring

We define the fusion ring $(\mathcal{T}_m, \oplus, \boxtimes)$ generated as the $\mathbb{Z}$-module, by the discrete series of $\mathfrak{Vir}_{1/2}$ at fixed charge c_m, with $m = \ell + 2$

Lemma 4.29. *(closure under fusion)*

(a) Each H_i is contains in some $H_\alpha^{\boxtimes n}$.

(b) The H_i's are closed under Connes fusion.

(c) $H_i \boxtimes H_j = \bigoplus m_{ij}^k H_k$ with $m_{ij}^k \in \mathbb{N}$

Proof. (a) Direct because α is a weak generator.
(b) Since $H_i \subset H_\alpha^{\boxtimes m}$ and $H_j \subset H_\alpha^{\boxtimes n}$ for some m, n, we have $H_i \boxtimes H_j \subset H_\alpha^{\boxtimes m+n}$, which is, by induction, a direct sum of some H_i. Now, by Schur's lemma any subrepresentations of a direct sum of irreducibles, is a direct sum of irreducibles; then, so is for $H_i \boxtimes H_j$.
(c) By induction, $H_\alpha^{\boxtimes m+n}$ admits only finite multiplicities. ☐

Definition 4.30. *(Quantum dimension) A quantum dimension is an application $d : \mathcal{T}_m \to \mathbb{R} \cup \{\infty\}$, which is additive and multiplicative for $\oplus$ and $\boxtimes$, and positive (possibly infinite) on the base (H_i).*

Reminder 4.31. *On a fusion ring, finite as $\mathbb{Z}$-module, the quantum dimension d is finite if $\forall A \in \mathcal{T}_m$, $\exists B \in \mathcal{T}_m$ such that $H_0 \leq A \boxtimes B$. If so, B is unique and called the dual of A, noted $A^\star$.*

Remark 4.32. $H_0 \leq H_\alpha \boxtimes H_\alpha$. *Then, H_α^ℓ is self-dual and $d(H_\alpha)$ finite.*

Corollary 4.33. *The quantum dimension is finite on the fusion ring.*

Proof. Because H_α is a weak generator, $\forall i$, $H_i \leq H_\alpha^{\boxtimes n}$ for some n, then $d(H_i) \leq d(H_\alpha)^n$ finite. $\square$

Reminder 4.34. *(Frobenius reciprocity) If $nA \leq B \boxtimes C$ then $nC \leq B^\star \boxtimes A$.*

Reminder 4.35. *(Perron-Frobenius theorem) An irreducible matrix with positive entries admits one and only one positive eigenvalues. The corresponding eigenspace is generated by a single vector $v = (v_i)$, with $v_i > 0$.*

Corollary 4.36. *A quantum dimension on $\mathcal{T}_m$ with $d(H_0) = 1$ is uniquely determined, and given by the fusion matrix of $H_\alpha = H_\alpha^\star$.*

Proof. $H_\alpha \boxtimes (\sum d(H_j)H_j) = \sum n_{\alpha j}^k d(H_j)H_k = \sum d(\sum n_{\alpha j}^k H_j)H_k$
$= \sum d(\sum n_{\alpha k}^j H_j)H_k = \sum d(H_\alpha H_k)H_k = d(H_\alpha)(\sum d(H_k)H_k)$.
Note that $n_{\alpha j}^k = n_{\alpha k}^j$ is immediate from Frobenius reciprocity and H_α self-dual. Next, α is a weak generator, so the fusion matrix M_α, is irreducible. The result follows with the Perron-Frobenius theorem, with $v_i = d(H_i)$. $\square$

4.6 The fusion ring and index of subfactor.

Definition 4.37. *Let* $\langle a, b\rangle_n = \{c = |a-b|, |a-b|+1, ..., a+b \mid a+b+c \leq n\}$.

Corollary 4.38. *(Connes fusion rules for α and β)*

$$(a) \quad H_\alpha^\ell \boxtimes H_{i'j'}^\ell = \bigoplus_{(i'',\, j'')\in\langle\frac{1}{2},i'\rangle_\ell\times\langle\frac{1}{2},j'\rangle_{\ell+2}} H_{i''j''}^\ell$$

$$(b) \quad H_\beta^\ell \boxtimes H_{i'j'}^\ell \leq \bigoplus_{(i'',\, j'')\in\langle 0,i'\rangle_\ell\times\langle 1,j'\rangle_{\ell+2}} H_{i''j''}^\ell$$

Proof. Immediate from theorem 3.47 and sections 4.3, 4.4. $\square$

Reminder 4.39. *(Connes fusion rules for $L\mathfrak{g}$ at level ℓ [25])*

$$H_i^\ell \boxtimes H_j^\ell = \bigoplus_{k\in\langle i,j\rangle_\ell} H_k^\ell$$

Reminder 4.40. *(Quantum dimension [25])*

$$d(H_i^\ell) = \tfrac{sin(p\pi/m)}{sin(\pi/m)}$$

with $m = \ell + 2$ and $p = dim(V_i) = 2i + 1$.

Definition 4.41. *Let $(\mathcal{R}_\ell, \oplus, \boxtimes)$ be the fusion ring generated as $\mathbb{Z}$-module by discrete series of $LSU(2)$ at level ℓ.*

Remark 4.42. *H^m_{pq} and $H^m_{m-p,m+2-q}$ are the same representation of $\mathfrak{Vir}_{1/2}$ because h^m_{pq} and $h^m_{m-p,m+2-q}$.*

Definition 4.43. *Let $\tilde{\mathcal{T}}_m$ be a formal associative fusion ring, generated by $(\tilde{H}^m_{pq})$ (or $(\tilde{H}^\ell_{ij})$ with the other notation), with every $\tilde{H}^m_{pq}$ distinct (in particular $\tilde{H}^m_{pq} \neq \tilde{H}^m_{m-p,m+2-q}$), using the fusion rules of corollary 4.38.*

Proposition 4.44. *The ring $\tilde{\mathcal{T}}_m$ is isomorphic to $\mathcal{R}_\ell \otimes_{\mathbb{Z}} \mathcal{R}_{\ell+2}$.*

Proof. Let the bijection $\varphi : \tilde{\mathcal{T}}_m \to \mathcal{R}_\ell \otimes_{\mathbb{Z}} \mathcal{R}_{\ell+2}$ with $\varphi(\tilde{H}^\ell_{ij}) = (H^\ell_i, H^\ell_j)$. The fusion matrix of $\tilde{H}^\ell_\alpha$ is clearly equal to the fusion matrix of $(H^\ell_{1/2}, H^{\ell+2}_{1/2})$. Then, by Perron-Frobenius theorem, $\tilde{H}^\ell_{ij}$ and (H^ℓ_i, H^ℓ_j) has the same quantum dimension. Now, $d(\tilde{H}^\ell_\beta).d(\tilde{H}^\ell_{i'j'}) \leq \sum d(\tilde{H}^\ell_{i''j''})$, and $d(H^\ell_0, H^\ell_1).d(H^\ell_{i'}, H^\ell_{j'}) = \sum d(H^\ell_{i''}, H^\ell_{j''})$. So, by positivity, the previous inequality is an equality and:

$$\tilde{H}^\ell_\beta \boxtimes \tilde{H}^\ell_{i'j'} = \bigoplus_{(i'',\, j'') \in \langle 0,i' \rangle_\ell \times \langle 1,j' \rangle_{\ell+2}} \tilde{H}^\ell_{i''j''}$$

So, the fusion rules for $\tilde{H}^\ell_\beta$ is also the same that for (H^ℓ_0, H^ℓ_1). Now, by associativity, the fusion rules for $\tilde{H}^\ell_\alpha$ and $\tilde{H}^\ell_\beta$ give all the fusion rules.
The result follows. □

Corollary 4.45. *$\mathcal{T}_m$ is isomorphic to the subring of $(\mathcal{R}_\ell \otimes_{\mathbb{Z}} \mathcal{R}_{\ell+2}) \otimes \mathbb{Q}$ generated by $\frac{1}{2}[(H^\ell_i, H^\ell_j) + (H^\ell_{\frac{\ell}{2}-i}, H^\ell_{\frac{\ell+2}{2}-j})]$; or to $(\mathcal{R}_\ell \otimes_{\mathbb{Z}} \mathcal{R}_{\ell+2})/((H^\ell_i, H^\ell_j) - (H^\ell_{\frac{\ell}{2}-i}, H^\ell_{\frac{\ell+2}{2}-j}))$. In particular, the fusion is commutative.*

Proof. Immediate. □

Theorem 4.46. *(Connes fusion for $\mathfrak{Vir}_{1/2}$)*

$$H^\ell_{ij} \boxtimes H^\ell_{i'j'} = \bigoplus_{(i'',\, j'') \in \langle i,i' \rangle_\ell \times \langle j,j' \rangle_{\ell+2}} H^\ell_{i''j''}$$

Proof. Immediate. □

Remark 4.47. *$H^\ell_{00} \leq (H^\ell_{ij})^{\boxtimes 2}$, so that H^ℓ_{ij} is self-dual.*

Theorem 4.48. *(Quantum dimension for $\mathfrak{Vir}_{1/2}$)*

$$d(H^\ell_{ij}) = d(H^\ell_i).d(H^{\ell+2}_j) = \tfrac{sin(p\pi/m)}{sin(\pi/m)}.\tfrac{sin(q\pi/(m+2))}{sin(\pi/(m+2))}$$

with $m = \ell + 2$, $p = 2i + 1$ and $q = 2j + 1$.

Proof. Immediate. □

Theorem 4.49. *(Jones-Wassermann subfactor)*

$$\pi^\ell_{ij}(\mathfrak{Vir}_{1/2}(I))'' \subset \pi^\ell_{ij}(\mathfrak{Vir}_{1/2}(I^c))^\natural$$

It's a finite depth, irreducible, hyperfinite III_1-subfactor, isomorphic to the hyperfinite III_1-factor $\mathcal{R}_\infty$ tensor the II_1-subfactor :

$$(\bigcup \mathbb{C} \otimes End_{\mathfrak{Vir}_{1/2}}(H^\ell_{ij})^{\boxtimes n})'' \subset (\bigcup End_{\mathfrak{Vir}_{1/2}}(H^\ell_{ij})^{\boxtimes n+1})'' \text{ of index } d(H^\ell_{ij})^2.$$

Proof. It's finite depth because there is only finitely many irreducible positive energy representations of charge c_m. Next, the hyperfinite III_1-subfactor and the irreducibility has already been proven before. The higher relative commutants can be calculated using the method of H. Wenzl [29]. The rest follows from the work of S. Popa [16]. □

References

[1] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.

[2] I. B. Frenkel, N. Yu. Reshetikhin, *Quantum affine algebras and holonomic difference equations.* Comm. Math. Phys. 146 (1992), no. 1, 1–60.

[3] P. Goddard, A. Kent, D. Olive, *Unitary representations of the Virasoro and super-Virasoro algebras.* Comm. Math. Phys. 103 (1986), no. 1, 105–119.

[4] R. Goodman, N. R. Wallach, *Projective unitary positive-energy representations of* $\mathrm{Diff}(S^1)$. J. Funct. Anal. 63 (1985), no. 3, 299–321.

[5] K. Iohara, Y. Koga, *Fusion algebras for* $N = 1$ *superconformal field theories through coinvariants. II.* $N = 1$ *super-Virasoro-symmetry.* J. Lie Theory 11 (2001), no. 2, 305–337

[6] V.F.R. Jones, *Index for subfactors.* Invent. Math. 72 (1983), no. 1, 1–25.

[7] V.F.R. Jones, *Fusion en algèbres de von Neumann et groupes de lacets (d'après A. Wassermann).*, Sminaire Bourbaki, Vol. 1994/95. Astérisque No. 237 (1996), Exp. No. 800, 5, 251–273.

[8] V.F.R. Jones, V. S. Sunder, *Introduction to subfactors.* London Mathematical Society Lecture Note Series, 234. Cambridge University Press, 1997.

[9] V. G. Kac, I. T. Todorov, *Superconformal current algebras and their unitary representations.* Comm. Math. Phys. 102 (1985), no. 2, 337–347.

[10] V. G. Kac, A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras.* Advanced Series in Mathematical Physics, 2. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987.

[11] T. Loke, *Operator algebras and conformal field theory for the discrete series representations of Diff*$(\mathbb{S}^1)$, thesis, Cambridge 1994.

[12] E. Nelson, *Analytic vectors.* Ann. of Math. (2) 70 1959 572–615.

[13] S. Palcoux, *Neveu-Schwarz and operators algebras I : Vertex operators superalgebras*, arXiv:1010.0078 (2010)

[14] S. Palcoux, *Neveu-Schwarz and operators algebras II : Unitary series and characters*, arXiv:1010.0077 (2010)

[15] V. Pasquier, H. Saleur, *Common structures between finite systems and conformal field theories through quantum groups.* Nuclear Phys. B 330 (1990), no. 2-3, 523–556.

[16] S. Popa, *Classification of subfactors and their endomorphisms.* CBMS Regional Conference Series in Mathematics, 86 , 1995.

[17] L. Pukanszky, *The Plancherel formula for the universal covering group of* SL(R, 2). Math. Ann. 156 1964 96–143.

[18] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. xv+361 pp

[19] V. Toledano Laredo, *Fusion of Positive Energy Representations of LSpin(2n)* , thesis, Cambridge 1997, arXiv:math/0409044 (2004)

[20] M. Takesaki, *Conditional expectations in von Neumann algebras.* J. Functional Analysis 9 (1972), 306–321.

[21] A. Tsuchiya, Y. Kanie, *Vertex operators in conformal field theory on* P^1 *and monodromy representations of braid group.* Conformal field theory and solvable lattice models (Kyoto, 1986), 297–372, Adv. Stud. Pure Math., 16, Academic Press, Boston, MA, 1988.

[22] A. Tsuchiya, T. Nakanishi, *Level-rank duality of WZW models in conformal field theory.* Comm. Math. Phys. 144 (1992), no. 2, 351–372.

[23] R. W. Verrill, *Positive energy representations of* $L^\sigma SU(2r)$ *and orbifold fusion.* thesis, Cambridge 2001.

[24] A. J. Wassermann, *Operator algebras and conformal field theory.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 966–979, Birkhuser, Basel, 1995.

[25] A. J. Wassermann, *Operator algebras and conformal field theory. III. Fusion of positive energy representations of* $\mathrm{LSU}(N)$ *using bounded operators.* Invent. Math. 133 (1998), no. 3, 467–538.

[26] A. J. Wassermann, *Kac-Moody and Virasoro algebras*, 1998, arXiv:1004.1287 (2010)

[27] A. J. Wassermann, *Subfactors and Connes fusion for twisted loop groups*, arXiv:1003.2292 (2010)

[28] A. J. Wassermann, private notes.

[29] H. Wenzl, *Hecke algebras of type* A_n *and subfactors.* Invent. Math. 92 (1988), no. 2, 349–383.

[30] R. J. Zimmer, *Ergodic theory and semisimple groups.* Monographs in Mathematics, 81. Birkhuser Verlag, Basel, 1984.